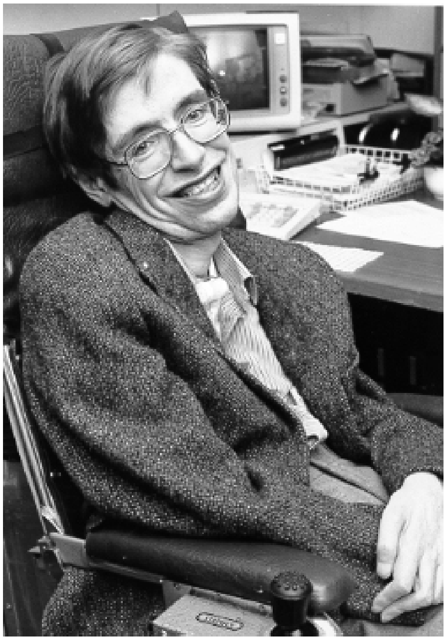
ÉCLAT
MATHEMATICS JOURNAL

Lady Shri Ram College For Women

Volume-IX
2017-2018



STEPHEN HAWKING

This journal is dedicated to Stephen Hawking, Fellow of Royal Society who was best known for his contributions to the fields of cosmology, general relativity and quantum gravity, especially in the context of black holes.
His work on the origins and structure of the universe, from the Big Bang to ground-breaking theorems regarding singularities within the framework of general relativity, making the theoretical prediction that black holes should emit radiation (known today as Hawking radiation) revolutionized the field.
Stephen Hawking will be continued to be remembered for his own theories, of cosmology in general and for how he brought science in the public gaze through his popular book *A Brief History of Time: From the Big Bang to Black Holes.*

PREFACE

Éclat, 2017 - 2018 has been an extraordinary journey at many levels. The journal, which was only an idea a few years back has become an integral part of the Mathematics department of the college. The eight papers that have been included in this year's edition have been very carefully written and compiled so as to stimulate the thought process of the readers.

Just like the previous year, Éclat 2017 - 2018 contains four categories of mathematics namely, History of mathematics, Rigour in mathematics, Inter disciplinary aspects of mathematics and Extension of course content. We have tried our best to make the journal inclusive so that readers from all fields can find a suitable content of their interest. The four categories discussed in the journal will give you a comprehensive idea about the evolution of the subject over the years. We hope that you will get a detailed knowledge of the various topics of mathematics.

The publication of this journal would have been impossible without the enthusiasm of the department of Mathematics of our college. We sincerely thank the faculty of the Department of Mathematics, Lady Shri Ram College for Women, for guiding and supporting us throughout the year. We are open to any suggestions, corrections and submission from our readers.

We dedicate this volume of the journal to Sir Stephen Hawking, Fellow of Royal Society who was best known for his contributions to the fields of cosmology, general relativity and quantum gravity, especially in the context of black holes.

The Editorial Team

Aakshi Malasi , B. Sc. (Hons.) Mathematics, 3rd Year
Nikita Sobti, B. Sc. (Hons.) Mathematics, 3rd Year
Kushagri Tandon, B. Sc. (Hons.) Mathematics, 2nd Year
Namrata Lathi, B. Sc. (Hons.) Mathematics, 2nd Year

CONTENTS

History of Mathematics

Mathematics is one of the oldest academic discipline involving stimulating and intriguing concepts.It is far beyond the ken of one individual and to make any contribution to the evolution of ideas, and understanding of the motivation behind the ideas is needed. The section covers the genesis of mathematical ideas, the stream of thought that created the problem and what led to its solution. The aim is to acquaint the readers with historically important mathematical vignettes and make them inured in some important ideas of Mathematics.

Cassini Ovals

Vandita Shankar

Abstract

This paper discusses Cassini ovals which were discovered by Domenico Cassini and were employed to study the relative motions between the Earth and Sun. The paper focusses on the polar and Cartesian equation of the Cassinian curve and its various applications in nuclear, solar and radar systems.

"There is geometry in the humming of strings, there is music in the spacing of spheres"

1 Giovanni Domenico Cassini

Astronomer and Mathematician Giovanni Domenico Cassini is associated with a number of scientific discoveries and projects, including the first observations of Saturn's moons. For this reason, the Cassini spacecraft that was launched in 1997 and plunged into the planet in 2017 was named after him.

Giovanni Domenico was born on **June 8, 1625**, in Perinaldo, Republic of Genoa (now Italy). He was the first of the well-known Cassini family of astronomers and is sometimes referred to as Cassini I. He was the first to observe four of Saturn's moons Iapetus (1671), Rhea (1672), Tethys (1684) and Dione(1684) which he called Sidera Lodoicea, or **Louisian Stars**, after the French king.

2 History of Cassini Ovals

The curve was first investigated by **Giovanni Cassini** in **1680** when he was studying the relative motions of the Earth and Sun. He believed that the Sun travelled round the Earth on one of these ovals, with the Earth at one focus of the oval.

1. Cassini assumed that the Earth is on one of the foci of the curve and the Sun describes it in its proper motion in such a way that if you draw two lines from the centre of the Sun to the two foci of the curve, the rectangle defined by these lines is always equal to the rectangle defined by the larger and the smaller distances between the Sun and the Earth.

Figure 1: Giovanni Domenico Cassini

2. The Cassinian curve/oval is now a purely geometric curve because it is known that planets describe Apollonian or ordinary ellipses. It is argued why Cassini substituted the Cassinian oval (ellipse) over Kepler's ellipse to study the relative motions between the Earth and Sun. However, it is known that most of the planets describe ellipses with low eccentricity and that in an ellipse with low eccentricity, the angular sectors created by vector rays from a focus are almost proportional to the corresponding angles at the other focus.

3. The ratio between the infinitely small sector and the corresponding angle is like the rectangle of two lines passing through the focus, and in an ellipse with low eccentricity. Since the ratio between elementary sectors with corresponding angles is like this rectangle, it will be constant in a curve where the rectangle is constant. Therefore, he imagined the cassinoid.

3 Introduction to Cassini Ovals

The Cassini ovals are the locus of a point P that moves so that the product of its distances from two fixed points S and T [in this case the points $(+a, 0)$ and $(-a, 0)$] is a constant c^2. The shape of the curve depends on $\frac{c}{a}$.

If $c > a$ then the curve consists of two loops. If $c < a$ then the curve consists of a single loop. The case where $c = a$ it produces a Lemniscate of Bernoulli (a figure of eight type curve introduced by **Jacob Bernoulli**).

Cassinian ovals are anallagmatic curves. They are defined by the bipolar equation $rr' = k^2$.

1. The Cassini ovals are defined in two-center bipolar coordinates by the equation:

$$r_1 r_2 = k^2$$

2. The Cassini ovals have the Cartesian equation:

$$[(x-a)^2 + y^2][(x+a)^2 + y^2] = b^4$$

3. The Cassini ovals have the polar equation:

$$r^4 + a^4 - 2a^2 r^2 [1 + \cos 2\theta] = b^4$$

4 Applications of Cassini Ovals

Cassinian curves have various scientific applications such as nuclear physics, biosciences, acoustics, etc. Besides being a model for the orbit of planets, the multi foci Cassini ovals add new dimensions to analytical geometry (coordinate geometry) and other subjects related to mathematics beyond the prevailing concept of ellipse.

1. **Formation of a Torus** - By rotating the two-dimensional Cassini oval when $\frac{\sqrt{2}}{2} < \frac{a}{b} < 1$ around the vertical axis we get a three-dimensional shape of an oblate disc sphere with a concavity on the top and bottom which resembles a torus. By changing the parameters a and b we get different shapes having different diameter, thickness and concavity. However, since those parameters are not directly related with the thickness or diameter the influence is an indirect one, therefore introducing a new parameter c we get a better flexibility for changing the three dimensional shape.

2. **Military and Commercial Purposes** - The unique geometrical properties derived from Cassinian Ovals are used for both military and commercial purposes. A few examples are new generation bistatic radar and sonar systems, modeling human red blood cells, simulating light scattering, modelling textile fabric, modelling population growths etc.

   (a) **Bistatic and Multistatic Radars** - Radar and sonar systems are categorized as monostatic, bistatic and multistatic systems. In both, bistatic and multistatic radar and sonar systems the shape of the detection zone of the system is modeled with the interior of Cassini ovals where the source, receiver and the target forms a bistatic triangle. The multistatic underwater sensor fields are optimized based on the shape and area of the ovals.

(b) **Nuclear Science** - The Cassini oval theory suggests that the electron orbit will vary with the changes in the shape of the nucleus.

It clubs the two great concepts; the concept of classical physics, where an electron behaves as a particle and moves in standard orbits around the nucleus; and the concept of quantum mechanics which indicates the wave property of the electron and also the probability of the electron clouds.

As another example to nuclear science, in the cross-sectional area of a nuclear magnetic resonance coil that used to induce electrically conductive coil is modelled as a Cassini oval.

(c) **Scattering of Light Particles** - Cassini ovals are a good fit for the simulation of light scattering by small particles. Although there are a great variety of methods to simulate the scattering which have been developed over the years, the null-field method with discrete sources is applied to analysis of light scattering by biconcave Cassini-like particles, which can be described as oblate disc spheres with central concavities on their top and bottom.

The appropriate mathematical description of the concave particle shapes that are used to simulate the light scattering is easily done by Cassini ovals.

(d) **Modelling of Evolutionary Processes** - Cassinian ovals are also used to model evolutionary processes such as morphogenetic sequences as well as to model the sub-atomic level and electro-magnetic activity in the case of wires of equal current and direction.

(e) **Optimization of Fuel Tanks** - For optimization of the fuel tanks used in Marshall Space Flight Centre, tank end closures are usually elliptically shaped. However, Cassinian domes may provide as much volume as an ellipse in a shorter length and with less discontinuity at the edges for a total vehicle net weight saving.

(f) **Modelling of Population Distribution** - The ovals are also used as a model of population distribution between two regions where the spatial structures are given.

Such an example is the growth of population between the two metropolitan cities, Beijing and Tianjin of China which is a distinct dual-nuclei metropolitan area in the world. It is modelled by the $\frac{a}{b}$ characteristic index of Cassini ovals. In the study when $\frac{a}{b} > 1$, the population density is more than 3000 persons/$km^2$and when $\frac{a}{b} = 1$, the population density is about 3000 persons/km^2.

5 Conclusion

Cassini ovals are related to ellipses but the former has product of the distances constant which makes it an interesting multi foci curve. Cassini ovals are now a purely geometric curve as the Kepler ellipses are utilised to study the relative motion between Sun and Earth.

This unique property enables these ovals to be utilized in various scientific, military and commercial areas to model phonemes. Therefore, it is important to give them greater importance so that more useful properties and analytical expressions can be derived.

References

[1] Gray, A., *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press, 1997

[2] Lawrence, J. D., *A Catalog of Special Plane Curves*, Dover Publications, 1972

[3] Lockwood, E. H., *A Book of Curves*, Cambridge University Press, 1961

[4] Karatas, M.(2013), *A multi foci closed curve: Cassini Oval, its properties and applications*, Dogus University, 14(2): 233-250

[4] www.2dcurves.com/quartic/quarticca.html

[5] www.history.mcs.st-andrews.ac.uk

[6] www.mathworld.wolfram.com

[7] www.encyclopediaofmath.org

VANDITA SHANKAR, B.Sc.(H) MATHEMATICS, 2nd SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI
vandita873@gmail.com

The Man who believed in the Eternal Beauty of Mathematics: G. H. Hardy

Richa Sharma

Abstract

This paper discusses the works of one of the greatest mathematicians ever, G. H. Hardy. Be it his number theory, mathematical analysis or the Hardy-Weinberg principle, the world can never forget his contributions to the field of pure mathematics and his ideology of believing in the eternal beauty lying in the subject of mathematics and the way he compared mathematics to art and poetry through his essay on the aesthetics of mathematics, *"A Mathematician's Apology"*.

1 Early Life and Education

Godfrey Harold Hardy was born on the 7th of February, 1877 in the village of Cranleigh, near Guildford, in the country of Surrey. Hardy displayed a special talent and interest in mathematics from an early age. When he was taken to church he passed the time by factorizing the numbers of the hymns.

Hardy chose Cambridge and particularly Trinity because of its mathematical traditions, but for anyone aiming at a career in research it had little to offer compared with what was available in Paris or Berlin. In particular the examination system was desperately in need of reform. He rightly considered the Tripos[1] an utter waste of time and tried changing his course from mathematics to history. However, as he writes in his memoir,

A Mathematician's Apology:

"My eyes were first opened by Professor Love, who taught me for a few terms and gave me my first serious conception of analysis. But the great debt I owe to him – he was, after all, primarily an applied mathematician and it was his advice to read Jordan's Cours d' Analyse: and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learned for the first time as I read it what mathematics really meant. From that time onwards, I was in my way a real mathematician with sound mathematical ambitions and a genuine passion for mathematics."

[1] the final honours examination for a BA degree at Cambridge University

Figure 1: G H Hardy

2 Later Life and Contributions

Later, as a fellow Hardy was finally free to devote his time to pure mathematics, and he did so with great enthusiasm and fervour. Over the next ten years he produced several papers, which established his reputation as an analyst, and wrote his classic textbook, **A Course of Pure Mathematics**.
In 1910, Hardy was elected as to the fellowship of the **Royal Society of London**. Despite what many saw as a highly productive period between the years 1900 and 1910, he himself felt that he did not publish as much of real value in that decade, and said so in the *Apology* as " I wrote a great deal in the next ten years but very little of any importance ; there are not more than five papers I can remember with some satisfaction ." He believed his best work came afterwards, out of his association with **Littlewood** and **Ramanujan**.
Hardy began his collaboration with Littlewood, who was eight years his junior, in 1911. The two men worked together mainly by correspondence, even when they were living in the same college. Meanwhile, Hardy was growing discontented with life in Cambridge and controversies arising from the First World War had much to do with this. At Oxford, he reached the peak of his career, setting up a flourishing research school. Hardy's unorthodox views were appreciated at New College, of which he automatically became a fellow. He spent 1928-29 as exchange professor at Princeton and the California Institute of Technology, while Veblen came to Oxford in his place. In 1913, Hardy returned to Cambridge as Sadleirian Professor of Pure Mathematics, and was invited to become a **fellow of Trinity** again. **The London Mathematical Society** occupied an important place in Hardy's life. He served on the council from 1905 to 1908 and from 1914 almost continuously until his final

retirement in 1945. He was one of the secretaries from 1917 to 1945 and twice the president, never missing a meeting.

In 1939, he suffered a coronary thrombosis, which put an end to sporting activities, such as cricket and real (or royal) tennis, that he enjoyed so much. When he died, on December 1, 1947, in his seventieth year, he was listening to his sister read to him from a book about the history of cricket at Cambridge. Hardy was the recipient of many honours, and from the Royal Society of London, first a **Royal medal**, then the **Sylvester medal** and finally the **Copley medal**, which he was due to receive on the day of his death.

3 Major Contributions

1. Hardy-Weinberg Law

This is an algebraic equation that describes the genetic equilibrium within a population. It was discovered independently in 1908 by Wilhelm Weinberg, a German physician, and Godfrey Harold Hardy.

- The science of population genetics is based on this principle, which may be stated as follows: in a large, random-mating population, the proportion of dominant and recessive genes present tends to remain constant from generation to generation unless outside forces act to change it.

- In such a way even the rarest forms of genes, which one would assume would disappear, are preserved. The outside forces that can disrupt this natural equilibrium are selection, mutation and migration.

- The law helped in affirming natural selection as the primary mechanism of evolution. Certain gene-controlled traits are selected for or selected against by the partners involved. Over a long period of time, this selective pressure will change the frequency of appearance of certain gene forms, and the traits they control will become commoner or rarer in the population.

- Medical geneticists can use the Hardy–Weinberg law to calculate the probability of human mating that may result in defective offspring. The law is also useful in determining whether the number of harmful mutations in a population is increasing as a result of radiation from industrial processes, medical techniques, and fallout.

2.Hardy-Ramanujan Asymptotic Formula

The Hardy-Ramanujan Asymptotic Partition Formula is stated as follows:
For a positive integer n, let $p(n)$ denote the number of unordered partitions of n, that is, unordered sequences of positive integers which sum to n; then the value of $p(n)$ is given asymptotically by -

$$p(n) \sim [1/(4n\sqrt{3})] \times (e^{\tau\sqrt{n/6}})$$

The value of $p(7)$ is 15. By $n = 70$, the number has rises to over 4 million; the Hardy-Ramanujan asymptotic estimate closely shadows this growth.Srinivasa Ramanujan (1887 - 1920) had already made deep discoveries about the partition function when he was himself 'discovered' by G.H. Hardy in 1913. Five years later, despite Ramanujan being already terminally ill, they published this remarkable asymptotic formula, derived from a series which yields the exact value of $p(n)$ in $\sqrt{n}$ terms.

3. Hardy-Littlewood Conjectures

(a) k-Tuple Conjecture

The first of the Hardy–Littlewood conjectures, the k-tuple conjecture states that the asymptotic number of prime constellations can be computed explicitly. In particular, unless there is a trivial divisibility condition that stops $p, p + a_1, ..., p + a_k$ from consisting of primes infinitely often, then such prime constellations will occur with an asymptotic density which is computable in terms of $a_1, ..., a_k$. Let $0 < m_1 < m_2 < ... < m_k$ then the k-tuple conjecture predicts that the number of primes p $\leq x$ such that $p + 2m_1, p + 2m_2, ..., p + 2m_k$ are all prime is :

$$\pi(m_1, m_2, ..., m_k)(x) \sim C(m_1, m_2, ..., m_k) \int_2^x dt/(ln^{(k+1)}t),$$

where,

$$C(m_1, m_2, ..., m_k) = 2^k \prod_q (1 - (w(q; m_1, m_2, ..., m_k))/q)/((1 - (1/q))^{(k+1)})$$

,

the product is over odd primes q and $w(q; m_1, m_2, ..., m_k)$ denotes the number of distinct residues of $0, m_1, ..., m_k (modq)$ (Halberstam and Richert 1974, Odlyzko *et al* 1999). If $k = 1$ then this becomes:

$$C(m) = 2 \prod_{q;qprime} (q(q-2))/((q-1)^2) \prod_{q|m} (q-1)/(q-2)$$

This conjecture is generally believed to be true, but has not been proven (Odlyzko *et al. 1999*).

(b)Second Hardy-Littlewood Conjecture

In number theory, the second Hardy–Littlewood conjecture concerns the number of primes in intervals. The conjecture states that:

$$\pi(x+y) \leq \pi(x) + \pi(y)$$

for $x, y \geq 2$, where $\pi(x)$ denotes the prime-counting function, giving the number of prime numbers up to and including x.

This means that the number of primes from $x+1$ to $x+y$ is always less than or equal to the number of primes from 1 to y. This was proved to be inconsistent with the first Hardy–Littlewood conjecture on prime k-tuples, and the first violation is expected to likely occur for very large values of x. For example, an admissible k-tuple (or prime constellation) of 447 primes can be found in an interval of $y = 3159$ integers, while $\pi(3159) = 446$. If the first Hardy–Littlewood conjecture holds, then the first such k-tuple is expected for x greater than $1.5 \times (10)^{174}$ but less than $2.2 \times (10)^{1198}$.

4. A Mathematician's Apology

" A Mathematician's Apology", first published in 1940 in England, is the memoir of the world-renowned mathematician, G.H Hardy, written in the last few years of his life while he was in failing health. The book is not mathematical; rather, it is an affirmation of a career that happens to be mathematical and purely speculative. A Mathematician's Apology is a lasting testament to Hardy's passion for intellectual pursuits. He likens mathematics to art and explains mathematics in much the same way a critic explains art. He elaborates on the qualities of mathematical genius and the logical reasons for pursuing a career in mathematics, and he briefly outlines three of the most basic and timeless theorems in order to illustrate the inherent beauty of mathematics for the layperson.

Many of the chapters also address the differences between theoretical or 'pure' mathematicsto which Hardy dedicated his life and several types of 'applied math', which he regards as largely inferior. The work also reveals the grave doubts Hardy harboured about the overall usefulness of his work and life.

4 Conclusion

Although Hardy was highly regarded in his home country, he was perhaps still more appreciated overseas, particularly in Germany. Hilbert thought of him as the best mathematician of England, particularly when he was collaborating with Littlewood. Hardy himself, with characteristic modesty, described Littlewood as the finest mathematician he had ever known; there was no-one else, he said, who could command such a combination of insight, technique, and power. It may seem surprising that there is not yet a full-scale biography of Hardy. Those who wish to know something of his special flavour may be recommended to read the *Apology*, which he wrote in 1940. This jewel of a book is still in print and has been translated into many languages.

References

[1] James, I., *Remarkable Mathematicians*, Cambridge University Press, 2002.

[2] Hardy, G.H., *A Mathematician's Apology*, Cambridge University Press, 1967.

[3] www.theoremoftheday.org/NumberTheory/Partitions/TotDPartitions.pdf

Further Reading

[1] www.encyclopedia.com/arts/culture-magazines/mathematicians-apology

[2] www.britannica.com/science/Hardy-Weinberg-law

RICHA SHARMA, B.Sc.(H) MATHEMATICS, 2nd SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI.
richa131999@gmail.com

Rigour in Mathematics

This section introduces advance Mathematics to the readers aiming at high standards of proofs. It stimulates interest and lays the foundation for further studies in different branches.

Regression Analysis

Tulika Rawat

Abstract

Regression analysis is a statistical process for estimating the relationship among variables .This paper deals with the properties and mathematical formulas of regression analysis and also various forms of regression.

1 Regression Analysis

Regression analysis is a mathematical measure of the average relationship between two or more variables in terms of the original units of the data.In regression analysis there are two types of variables:

1. The variable whose value is influenced or is to be predicted is called dependent variable. Dependent variable is also known as **regressed or explained variable**.
2. The variable which influences the values or is used for prediction is called **independent variable or regressor or pedictor or explanatory variable**.

1.1 Linear Regression

If the variable in a bivariate distribution are related then the points in the scatter diagram will cluster around some curve which is known as the **Curve Of Regression** here the regression between the variables is said to be curvilinear.
If the curve is a straight line,it is called the line of regression and the variables are said to possess **Linear Regression**.
The line of regression is the line which gives the best estimate to the value of one variable for any specific value of other variable.Thus the line of regression is called the line of **Best Fit** and is obtained by the principle of least squares.

1.2 Karl Pearson's Coefficient Of Correlation

The degree of linear relationship between two random variables X and Y is measured by a formula called correlation coefficient which is usually denoted by $r(X, Y)$.

1.3 Lines Of Regression

In order to solve the problem of predicting ,for an individual, the value of one variable(suppose y)from the given value of another variable (suppose x),we need to express the relationship between y and x in mathematical form.
Suppose in a particular case the approximate relation may be represented by a line:

$$Y = a + bX$$

To get an appropriate line ,it is necessary to determine a and b from the observed data. Let us assume that there are n given pairs of values of x and y, the i^{th} pair being denoted by (x_i,y_i).The above line gives as an estimate of x_i the value

$$y_i = a + bx_i(say)$$

The most satisfactory method of determining a and b would be the method of least squares, which consists in minimising the sum of squares of the errors of estimation. Therefore, we must choose a and b in such a way so that,

$$\begin{aligned} S^2 &= \sum_i (y_i - Y_i)^2 \\ &= \sum_i (y_i - a - x_i)^2 \end{aligned}$$

where,$(y_i - Y_i)$ is the error of estimate for the i^{th} pair,$i = 1, 2, 3, ..., n$. Here we are using the method of least squares because since the line is to be used for estimating purposes, it is reasonable to require that a and b such that these errors of estimate are as small as possible.
As we know,

$$S^2 = \sum_i (y_i - Y_i)^2$$

$$= \sum_i (y_i - a - x_i)^2$$

$$\frac{\delta S^2}{\delta a} \geq 0$$

$$\frac{\delta S^2}{\delta b} = 0$$

or

$$\sum_i (y_i - a - bx_i) = 0$$

and multiplying on both sides by x_i we get

$$\sum_i x_i(y_i - a - bx_i) = 0$$

i.e. on summing over x_i we get

$$\sum_i y_i = na + b\sum_i x_i$$

and again multiplying x_i on both sides

$$\sum_i (x_i y_i) = a\sum_i x_i + b\sum_i x_i^2$$

the roots of the equation are:

$$b = \frac{n\sum_i (x_i)(y_i) - (\sum_i x_i)(\sum_i y_i)}{(n\sum_i (x_i)^2) - (\sum_i x_i)^2}$$

$$= \frac{\sum_i (x_i y_i)/n - (\bar{x})(\bar{y})}{\sum_i (x_i)^2 - (\bar{x})^2}$$

$$= \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})/n}{\sum_i (x_i - \bar{x})^2/n}$$

$$= r\frac{\sigma_y}{\sigma_x}$$

$$a = \overline{y} - b\overline{x}$$

On substituting these values in equation $y = a + bx$,we get,

$$y = \overline{y} + r\frac{\sigma_y}{\sigma_x}(x - \overline{x})$$

This is the line of regression of y on x

$$a = \bar{y} - r\frac{\sigma_y}{\sigma_x}$$

$$= \quad y - intercept$$

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = Slope$$

b is the amount by which the predicted value of y increases for a unit increment in the value of x.

b=regression coefficient of y on x

Similarly,

$$x = \overline{x} + r\frac{\sigma_x}{\sigma_y}(y - \overline{y})$$

This is the line of regression of x on y, where $r\frac{\sigma_x}{\sigma_y} = \frac{\sigma_y}{r\sigma_x}b_{xy}$

1.4 Properties Of Regression Coefficients

1. If one of the regression is greater than unity, the other must be less than unity

$$b_{yx} > 1 \Rightarrow \frac{1}{b_{yx}} < 1$$

$$r^2 \leq 1 \Rightarrow b_{yx}.b_{xy} \leq 1$$

2. Correlation coefficient is the geometric mean between the regression coefficients

$$r = \pm\sqrt{b_{yx} * b_{xy}}$$

The sign of correlation coefficient must be similar to that of regression coefficients.

3. The modulus value of the arithmetic mean of the regression coefficients is not less than the modulus value of the correlation coefficients r.

$$|\frac{1}{2}(b_{yx} + b_{xy})| > |r|$$

1.5 Angle Between Two Lines Of Regression

- The slopes of the lines are $r\frac{\sigma_y}{\sigma_y}$ and $\frac{\sigma_y}{r\sigma_x}$
- θ is the acute angle between the two lines regression

$$\tan\theta = \left|\frac{r\frac{\sigma_y}{\sigma_x} - \frac{\sigma_y}{r\sigma_x}}{1 + r\frac{\sigma_y}{\sigma_x}\frac{\sigma_y}{r\sigma_x}}\right|$$

$$\tan\theta = \left|\frac{r^2 \approx 1}{r}\right| \left(\frac{\sigma_x.\sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

2 Curvilinear Regression

Sometimes the variability of Y is explained in a more better way using curvilinear regression rather than linear regression. Therefore, we fit the regression line using the formula

$$Y = a + b_1X + b_2X^2$$

Here by using the principles of the least squares we determine a, b_1 and b_2 to constant such that the E is minimum i.e.

$$E = \sum_{i=1}^{n}(y_i - a - b_1x_i - b_2x_i^2)^2$$

Equating the partial derivatives of E with respect to a,b and c separately, we get the normal equation for estimating a, b and c as

$$\frac{\delta E}{\delta a} = 0 = -2\sum_i(y_i - a - b_1x_i - b_2x_i^2)$$

$$\frac{\delta E}{\delta b_1} = 0 = -2\sum_i x_i(y_i - a - b_1x_i - b_2x_i^2)$$

$$\frac{\delta E}{\delta b_2} = 0 = -2\sum_i x_i^2(y_i - a - b_1x_i - b_2x_i^2)$$

$$\sum y_i = na + b_1\sum x_i + b_2\sum x_i^2$$

$$\sum x_iy_i = a\sum x_i + b_1\sum x_i^2 + b_2\sum x_i^3$$

$$\sum x_i^2y_i = a\sum x_i^2 + b_1\sum x_i^3 + b_2\sum x_i^4$$

On solving this for a, b_1 and b_2 we get the parabola of best fit.

3 Multiple Regression

Whenever we are interested in studying the joint effect of a group of variables upon a variable not included in that group, our study is that of **multiple regression**.

1. As in a bivariate data, here too, it may be that one of the p variables, say x_1, is of primary interest to us and we consider $x_2, x_3, \ldots, x_p$ together with x_1, in view of their possible influence on the latter.
2. The object is to establish a relationship between **dependant variable**, x_1,and **independant variables**,$x_2, x_3, \ldots, x_p$, with the idea of using this relationship for predicting the value of the regressand from a knowledge of the values of the regressors.

Let us assume that the relationship between x_1 and $x_2, x_3, \ldots, x_p$ is given by the following equation

$$x_1 = a + b_2x_2 + b_3x_3 + \ldots + b_px_p$$

• Here our data consists of p values , corresponding to the p variables, for each n individuals. The values of the variables for the α^{th} individual may be denoted by $x_{1\alpha}, x_{2\alpha}, \ldots, x_{p\alpha}$, where $\alpha = 1, 2, \ldots, n$.
• In order to determine the constants a, $b_2, b_3, b_4, \ldots, b_p$ on the basis of the data , we again make use of the **least square method**.
• We denote by $x_{1.23\ldots p}$ the difference between x_1 and $x_{1.23\ldots p} = a + b_2x_2 + b_3x_3 + \ldots + b_px_p$, then the error of estimation corresponding to the α^{th} individual is $x_{1.23\ldots p,\alpha}$.
The **least square method** means that the constant a, $b_2, b_3, \ldots, b_p$ are to be so determined that

$$\sum_\alpha x^2_{1.23\ldots p,\alpha} = \sum_\alpha (x_{1\alpha} - a - b_2x_{2\alpha} - \ldots - b_px_{p\alpha})^2$$

is minimum.

4 Example

Following records are available:
Regression equations:
$8x - 10y + 66 = 0$,
$40x - 18y = 214$,
σ_x=3.
Find the

1. mean values of x and y.
2. σ_y
3. coefficient of correlation

Solution
Since both the lines of regression pass through the point $(\bar{x}, \bar{y})$, we have

$$8\bar{x} - 10\bar{y} + 66 = 0$$

$$40\bar{x} - 18\bar{y} - 214 = 0$$

Solving them, we have

$$\bar{x} = 13$$

and

$$\bar{y} = 17$$

Writing the regression lines as

$$y = 0.8x + 6.6$$

and

$$x = 0.45y + 5.35,$$

We get regression coefficients

$$r\frac{\sigma_y}{\sigma_x} = 0.8$$

and

$$r\frac{\sigma_x}{\sigma_y} = 0.45$$

Multiplying them, $r^2 = 0.8 * 0.45 = 0.364$ or r=0.6.
Also,

$$r\frac{\sigma_y}{\sigma_x} = 0.8,$$

$$\frac{0.6 * \sigma_y}{3} = 0.8,$$

$$\sigma_y = 4$$

5 Uses of regression analysis

1. A pharmaceutical company used regression to assess the stability of the active ingredient in a drug to predict its shelf life and identify a suitable expiration date for the drug.
2. A credit card company applied regression analysis to predict monthly gift card sales and improve yearly revenue projections.
3. An insurance company used regression to determine the likelihood of a true problem existing when a home insurance claim was filed, in order to discourage customers from filing excessive or petty claims.
4. Regression Analysis can bring a scientific angle to the management of any businesses by reducing tremendous amount of raw data into actionable information, regression analysis leads the way to smarter and more accurate decisions.

References

[1] Gupta,S.C, and Kapoor,V.K, *Fundamentals Of Mathematical Statistics*, Sultan Chand and Sons, 2015

[2] Miller,I, and Miller,M, *John E Freund's The Fundamentals Of Mathematical Statistics*,Pearson, 2014

TULIKA RAWAT , B.Sc(H) MATHEMATICS , 6^{th} SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI
tulika730.lko@gmail.com

Applicability of Three Body Problem in Celestial Mechanics

Dr. Bhavneet Kaur

1 Introduction

The three-body problem is one of the central problems in field of Celestial Mechanics. It has many applications in different scientific areas, in particular, in the fields of Astrophysics and Astrodynamics. The problem is classified in two classes: the first one is the general problem which describes the motion of three celestial bodies under their mutual gravitational attraction. The second class is the restricted problem in which the third body has an infinitesimal mass compared with masses of the other two bodies and consequently it does not affect their motion.

One of the main applications of the general problem in Astrophysics is for instance the dynamics of triple stars systems. In the second half of the 20th century and even today. the study of scientific community has center its attention on the restricted three-body problem and there are a big number of papers studying different aspect of this problem. For instance, considering the influences of perturbed forces such as oblateness, radiation pressure, Coriolis and centrifugal forces, variation of masses, the Pointing Robertson effect, the atmospheric drag, the solar wind etc. Significant studies related with the libration points considering the oblateness of one or both primaries when the equatorial plane is coincident with the plane of motion are done by [Subbarao and Sharma (1975)], [Sharma and Subbarao (1978)] and [Markellos et al. (1996)].

Some works studying different aspects of the dynamics of the restricted problem when the three participating bodies are oblate spheroids are given by [El-Shaboury and El-Tantawy (1993)], [Abouelmagd and El-Shaboury (2012)] and [Elipe and Ferrer (1985)]. Interesting papers when one or both primaries are triaxial bodies are for instance [El-Shaboury et al. (1991)], [Khanna and Bhatnagar (1999)] and [Sharma and Bhatnagar (2001)]. Several authors have been devoted their efforts to study the effects of small perturbations in centrifugal and Coriolis forces as [Szebehely(1967)], [Bhatnagar & Hallan(1978)], [Devi and Singh (1994)] and [Shu et al. (2005)]. The existence and the linear stability of the libration points in the restricted problem for perturbed potentials between the bodies in the cases: the bigger primary is an oblate spheroid, both primaries are oblate spheroids, the primaries are spherical and the biggest primary is a source of radiation were studied in by B[Bhatnagar & Hallan(1979)]. They observed that the collinear points are unstable, the range of stability for the triangular points increases or decreases depending on the sign of a parameter which depends on the perturbed functions. Furthermore, [Singh and Ishwar (1999)] studies the effect of the oblateness and the radiation pressure at the locations of the triangular points and their linear stability when both the primaries are oblate and radiating. [Ishwar and Elipe (2001)] found the secular solutions at the triangular points in the generalized photogravitational restricted three-body problem.

The problem is generalized in the sense that the bigger primary is a source of radiation and the smaller one is an oblate spheroid. Moreover, [Abouelmagd et al. (2014a)] found the secular solution around the triangular equilibrium points and reduce it to a periodic solution in the frame work of the generalized restricted three-body problem, in sense that both primaries are oblate and radiating as well as the gravitational potential from a belt. They also showed that the linearized equation of motion of the infinitesimal body around the triangular equilibrium points has a secular solution when the value of the masses ratio is equal to the critical mass value. Numerical and graphical analysis in order to understand the effects of the perturbed forces are stated. [Mittal and Bhatnagar (2009)] studies the periodic orbits generated by Lagrangian solutions of the restricted three-body problem when the bigger body is a source of radiationand the smaller is an oblate spheroid. It is used the definition of [Karimov and Sokolsky (1989)] for mobile coordinates to determine these orbits and the predictor method to draw them. [Singh and Begha (2011)] studied the existence of periodic orbits around the triangular points in the restricted three body problem when the bigger primary is triaxial and the smaller one is considered as an oblate spheroid. In the range of linear stability under the effects of the perturbed forces of Coriolis and centrifugal, it is deduced that long and short periodic orbits exist around these points and are stated their periods, orientation and eccentricities affected by the non sphericity and the perturbations in the Coriolis and centrifugal forces. [Abouelmagd (2012)] studied the effects of oblateness coefficients J_2 and J_4 of the bigger primary in the planar restricted three-body problem on the locations of the triangular points and their linear stability. Recently, there are also some interesting papers connected with the restricted three-body problem, see [Beevi and Sharma (2012)], [Singh (2012)], [Singh and Taura (2013)], [Kishor and Kushvah (2013)],[Abouelmagd et al. (2013)], [Abouelmagd (2013)] and [Abouelmagd et al. (2014b)]. In last decades a great number of authors have studied the restricted three-body problem taking account the effects of oblateness of one or both primaries up to 10^{-3} of the main terms of the potential.

We now discuss about the Restricted three-body problem.

1.1 Restricted three-body problem (or Restricted Problem)

Statement: Two bodies revolve around their centre of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of three bodies is to describe the motion of the third body.

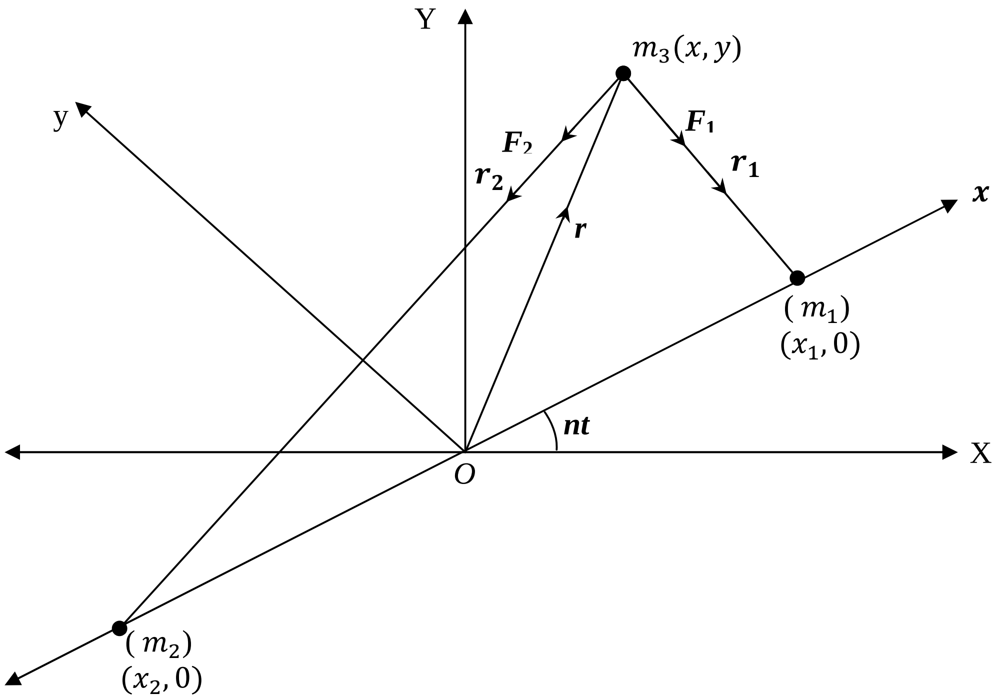


Figure 1: Configuration of the Restricted Three-Body Problem

In case, the primaries move in the elliptical orbits instead of circular orbits, the problem is known as **elliptical restricted three-body problem** and in case the third body is not moving in the plane of motion of the primaries, the problem is called the **three dimensional restricted three-body problem**.

The restricted three-body problem has in the past played an essential role in many different areas of dynamical astronomy and indications are that this will con-

tinue. The applications of the restricted problem to the Celestial mechanics form the basis of some lunar and planetary theories. The modern applications to space mechanics are probably even more cogent if not more numerous than the classical applications. The implications of the restricted problem for cosmology and stellar dynamics are also numerous. Many problems in space dynamics are of considerable importance and interest today. Many of these problems are new, and in what follows one of them will be contrasted to a classical problem.
Consider the famous classical three-body problem, the sun-earth-moon combination and the determination of the motion of the moon. We might think of two large bodies, the sun and the earth, which move around each other in approximate circles, and in their field a third body, the moon which moves on an approximate ellipse. This configuration is stationary in a sense, since no collisions take place.This is also true for the motion of a Trojan asteroid under the continued influence of Sun and Jupiter. On the other hand, one of the central problems in space science is to create artificial bodies. Sometimes collision orbits are desired. Problems with close approaches and collisions were hardly even treated in Celestial mechanics and these problems became important in the new science of space dynamics.

Euler was the first to contribute towards the restricted problem in 1772 in connection with his Lunar Theories. His main contribution was the introduction of a synodic (rotating) coordinate system resulting in what is called the Jacobi integral which as discovered by [Jacobi(1836)]. Implications of this integral are numerous. It determines the regions of motion. Its application to Celestial Mechanics was first made by [Hill (1878)]. [Poincaré(1899)] and [Birkhoff (1915)] are the pioneers in the qualitative methods of dynamics. Poincaré's famous work in three volumes 'Methodes Nouvelles' completed in 1899 was so new and original that many of its implications are still not clear.

Lagrange also mentioned the special case of the restricted problem when the Coriolis forces are omitted and a condition of symmetry regarding the masses of the primaries is introduced, the restricted problem may be solved in terms of elliptic functions. Prior to the lunar theory, in 1760, Euler gave the solution of the problem of two fixed centres of force, in which two fixed masses act on the third body according to Newton's law of gravitation. This is a special case of the restricted problem when both the Coriolis and the centrifugal forces are omitted. Euler's solution of the problem of two fixed centres of force can be used in connection with the artificial satellite problem, and as a reference orbit for the calculation of perturbations in the restricted problem.This study has paved way for a lot of research in the field of Celestial Mechanics especially n-body problem.

References

[Abouelmagd et al. (2013)] Abouelmagd, E.I., Asiri, H.M., Sharaf, M.A., 2013. The effect of oblateness in the perturbed restricted three-body problem. Meccanica 48, 24792490.

[Abouelmagd (2013)] Abouelmagd, E.I., 2013. The effect of photogravitational force and oblateness in the perturbed restricted three-body problem. Astrophysics and Space Science 346, 5169.

[Abouelmagd (2012)] Abouelmagd, E.I., 2012. Existence and stability of triangular points in the restricted three-body problem with numerical applications. Astrophysics and Space Science 342, 4553.

[Abouelmagd and El-Shaboury (2012)] Abouelmagd, E.I., El-Shaboury, S.M., 2012. Periodic orbits under combined effects of oblateness and radiation in the restricted problem of three bodies. Astrophysics and Space Science 341, 331341.

[Abouelmagd et al. (2014a)] Abouelmagd, E.I., Awad, M.E., Elzayat, E.M.A., Abbas, I.A., 2014a. Reduction the secular solution to periodic solution in the generalized restricted three-body problem. Astrophysics and Space Science 350, 495505.

[Abouelmagd et al. (2014b)] Abouelmagd, E.I., Guirao, J.L.G., Mostafa, A., 2014b. Numerical integration of the restricted three-body problem with Lie series. Astrophysics and Space Science 354, 369378.

[Beevi and Sharma (2012)] Beevi, A.S., Sharma, R.K., 2012. Oblateness effect of Saturn on periodic orbits in the SaturnTitan restricted three-body problem. Astrophysics and Space Science 340,245261.

[Bhatnagar & Hallan(1978)] Bhatnagar, K. B. and Hallan, P.P.:1978, 'Effect of Perturbations in Coriolis and Centrifugal Forces on the Stability of Libration Points in the Restricted Problem', Celestial Mechanics 18,105-112.

[Bhatnagar & Hallan(1979)] Bhatnagar, K.B., Hallan, P.P., 1979. Effect of perturbed potentials on the stability of libration points in the restricted problem. CeMec 20, 95 103.

[Birkhoff (1915)] Birkhoff, G. The restricted problem of three bodies. Rendiconti del Circolo Matematico di Palermo, 39(1):265-334, 1915.

[Devi and Singh (1994)] Devi, G.S., Singh, R., 1994. Location of equilibrium points in the perturbed photogravitational circular restricted problem of three bodies. BASI 22, 433437.

[Elipe and Ferrer (1985)] Elipe, A., Ferrer, S., 1985. On the equilibrium solution in the circular planar restricted three rigid bodies problem. CeMec 37, 5970.

[El-Shaboury et al. (1991)] El-Shaboury, S.M., Shaker, M.O., El-Dessoky, A.E., El Tantawy, M.A., 1991. The libration points of axisymmetric satellite in the gravitational field of two triaxial rigid body. EM&P 52, 6981.

[El-Shaboury and El-Tantawy (1993)] El-Shaboury, S.M., El-Tantawy, M.A., 1993. Eulerian libration points of restricted problem of three oblate spheroids. EM&P 63, 2328.

[Hill (1878)] Hill, G. W. Researches in the lunar theory. American Journal of Mathematics, 1(1):pp. 5-26, 1878.

[Jacobi(1836)] Jacobi, C. G. J. Sur le movement d'un point et sur un cas particulier du probléme des trois corps. Comptes Rendus de l'Académie des Sciences de Parisy, 3:59-61, 1836.

[Khanna and Bhatnagar (1999)] Khanna, M., Bhatnagar, K.B., 1999. Existence and stability of libration points in the restricted three body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid. IJPAM 30 (7), 721733.

[Ishwar and Elipe (2001)] Ishwar, B., Elipe, A., 2001. Secular solution at triangular equilibrium points in the generalized photogravitational restricted three body problem. Astrophysics and Space Science 277, 437444.

[Karimov and Sokolsky (1989)] Karimov, S.R., Sokolsky, A.G., 1989. The periodic motions generated by Lagrangian solutions of circular restricted three-body problem. CeMec 46, 335381.

[Kishor and Kushvah (2013)] Kishor, R., Kushvah, B.S., 2013. Periodic orbits in the generalized photogravitational Chermnykh-like problem with power-law profile. Astrophysics and Space Science 344, 333346.

[Markellos et al. (1996)] Markellos, V.V., Papadakis, K.E., Perdios, E.A., 1996. Non-linear stability zones around triangular equilibria in the plane circular restricted three-body problem with oblateness. Astrophysics and Space Science 245, 157164.

[Mittal and Bhatnagar (2009)] Mittal, A., Bhatnagar, K.B., 2009. Periodic orbits in the photogravitational restricted problem with the smaller primary an oblate body. Astrophysics and Space Science 323, 6573.

[Poincaré(1899)] Poincaré, H. and Magini, R. Les méthodes nouvelles de la mécanique céleste. IlNuovo Cimento Series 4, 10(1):128-130, 1899.

[Sharma and Bhatnagar (2001)] Sharma, R.K., Bhatnagar, K.B., 2001. Existence of libration points in the restricted three body problem when both primaries are triaxial rigid bodies. Indian J. Pure Appl. Math. 32 (1), 125141.

[Subbarao and Sharma (1975)] Subbarao, P.V., Sharma, R.K., 1975. A note on the stability of the triangular points of equilibrium in the restricted three-body problem. A&A 43, 381383.

[Sharma and Subbarao (1978)] Sharma, R.K., Subbarao, P.V., 1978. A case of commensurability induced by oblateness. CeMec 18, 185194.

[Shu et al. (2005)] Shu, Sh., Lu, Bk., Chen, Ws., Liu, Fy., 2005. Effect of perturbation of Coriolis and Centrifugal forces on the location and linear stability of the libration points in the Robe problem. ChA&A 29, 412429.

[Singh and Begha (2011)] Singh, J., Begha, J.M., 2011. Periodic orbits in the generalized perturbed restricted three-body problem. Astrophysics and Space Science 332, 319324.

[Singh (2012)] Singh, J., 2012. Motion around the out-of-plane equilibrium points of the perturbed restricted three-body problem. Astrophysics and Space Science 332, 303308.

[Singh and Taura (2013)] Singh, J., Taura, J.J., 2013. Motion in the generalized restricted threebody problem. Astrophysics and Space Science 343, 95106.

[Singh and Ishwar (1999)] Singh, J., Ishwar, B., 1999. Stability of triangular points in the photogravitational restricted three body problem. ChA&A 27, 415424.

[Szebehely(1967)] Szebehely, V.S.:1967 'Theory of orbits ', Academic Press, New York.

DR. BHAVNEET KAUR, ASSISTANT PROFESSOR, DEPARTMENT OF MATHEMATICS, LADY SHRI RAM COLLEGE FOR WOMEN, UNIVERSITY OF DELHI, DELHI bhavneet.lsr@gmail.com

Extension of Course Contents

A great deal of learning happens beyond the formal coursework. This section hence aims to provide a creative, fertile setting for productive research that goes beyond the confines of classroom, and precincts of syllabi. It strengthens and expands the existing knowledge and adds interests to the course and provides an experience of trans-formative learning.

Symmetry Methods to Solve First Order ODEs: Introduction to Lie Groups and Lie Symmetries

Kushagri Tandon

Abstract

This paper will be exploring the symmetry methods used in solving ordinary differential equations with emphasis on Lie symmetry groups. Symmetries can be used to determine a canonical coordinate system for a given differential equation. This paper will include an introduction to symmetries, introduction to lie groups and application of lie point symmetries in solving first order ordinary differential equations.

1 Introduction

Differential equations are used to model numerous phenomenon in the world. Study of differential equations plays a vital role in the physical sciences. These equations are often non-linear and solving them requires unique and creative methods. There are many ingenious techniques for obtaining exact solutions of differential equations, which work only for a very limited class of problems.

Sophus Lie, a Norwegian mathematician born in 1842, was the first to discover the relationship between group theory and traditional methods for finding solution curves.

He discovered way of using symmetries and groups of point transformatios to solve differential equations.

It is often quite easy to find symmetries of a given differential equation (even an unfamiliar one) and to use them systematically to obtain exact solutions. Symmetries can also be used to simplify problems and to understand bifurcations of nonlinear systems.

1.1 Symmetries

A symmetry is a rigid mapping from an object to itself or another object. It must preserve the structural properties of the original item. These mappings include rotations, translations, and reflections.

Example, $\mathbf{S_3}$, the group of symmetries of a triangle.

Symmetries preserve structure. In other words, they must satisfy the symmetry condition.

In case of differential equations, satisfying the symmetry condition requires that a symmetry maps one solution curve to another. The resulting solution curve must also satisfy the original differential equation.

To illustrate symmetry in differential equations, *consider the ordinary differential equation*

$$\frac{dy}{dx} = 0 \qquad \text{(Eq. 1.1)}$$

The set of all solutions of the ODE is the set of lines $y(x) = c,\ c \in \mathbb{R}$, which fills the (x, y) plane. This solutions is represented by horizontal lines, seen in Figure 1.1.

Lie symmetries are point transformations that map a point on a solution curve in $\mathbb{R}^2$ to another point on a solution curve in $\mathbb{R}^2$.

For a parameter $\lambda \in \mathbb{R}$, a **Lie symmetry** is given by,

$$P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y}) \qquad \text{(Eq. 1.2)}$$

An example of symmetry of the equation (Eq. 1.1) is,

$$(\hat{x}, \hat{y}) = (x, y + \lambda).$$

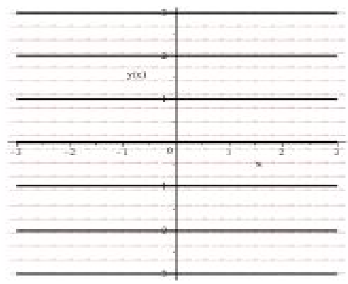


Figure 1.1: Solutions of equation (Eq. 1.1)

This symmetry will map a point (x, y) on one solution curve to a point $(\hat{x}, \hat{y})$ on another.

A *trivial symmetry* maps every solution curve to itself. For instance, a trivial symmetry of the equation (Eq. 1.1) is,

$$(\hat{x}, \hat{y}) = (x + \lambda, y)$$

This symmetry does not map to another solution curve, because the point $(\hat{x}, \hat{y})$ is on the same solution curve as (x, y). An equilateral triangle has a finite set of symmetries. Many objects have an infinite set of symmetries. For example, the (rigid) unit circle $x^2 + y^2 = 1$ has a symmetry,

$$\Gamma_\varepsilon : (x, y) \mapsto (\hat{x}, \hat{y}) = (x \cos\varepsilon - y \sin\varepsilon, x \sin\varepsilon + y \cos\varepsilon),\ \forall\varepsilon \in (-\pi, \pi] \qquad \text{(Eq. 1.3)}$$

In terms of polar coordinates, $\Gamma_\varepsilon : (\cos\theta, \sin\theta) \mapsto (\cos(\theta + \varepsilon), \sin(\theta + \varepsilon))$ as shown in Figure 1.2. Hence the transformation is a rotation by ε about the centre of the circle. It preserves the structure, and it is smooth and invertible (the inverse of a rotation by ε is a rotation by $-\varepsilon$. Note that $\hat{x}^2 + \hat{y}^2 = x^2 + y^2 = 1$

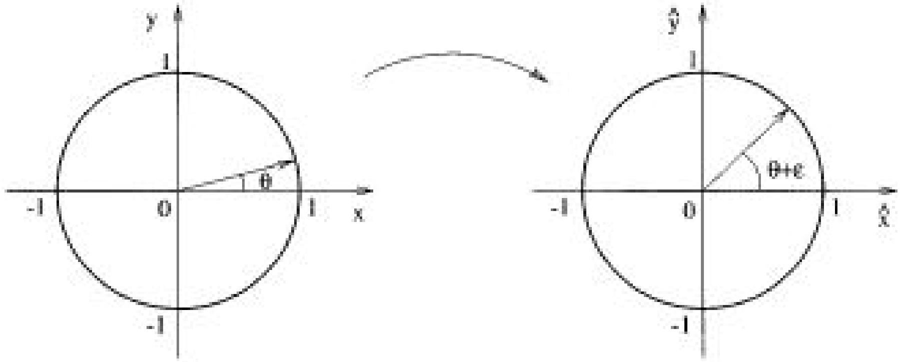


Figure 1.2: Rotation of the unit circle.

1.2 Lie Groups

A **Lie group** is a group of symmetries with a parameter $\lambda \in \mathbb{R}$. Lie group symmetries are the functions P_λ from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Let A be a set of points $(x, y) \in \mathbb{R}^2$ and let B be a set of points $(\hat{x}, \hat{y}) \in \mathbb{R}^2$.

A function $\mathbf{P}_\lambda$ maps A to B:

$$P_\lambda : A \mapsto B$$

Here, $\hat{x} = f(x, y, \lambda)$ and $\hat{y} = g(x, y, \lambda)$

The identity, I in a Lie symmetry group maps a point to itself. A set of symmetries forms a **Lie group** if it meets the following conditions:

1. P_λ is a bijection i.e. P_λ is a symmetry
2. $P_{\lambda_1} \circ P_{\lambda_2} = P_{\lambda_1+\lambda_2}$ i.e. the set of symmetries is closed under composition of mappings.
3. $P_0 = I$ is the identity of the set of symmetries.
4. $\forall \lambda_1 \in \mathbb{R}, \exists \lambda_2 = -\lambda_1$ s.t. $P_{\lambda_1} \circ P_{\lambda_2} = P_{\lambda_2} \circ P_{\lambda_1} = P_0 = I$ i.e. each point symmetry has an inverse in the set of symmetries.

The infinite set of symmetries Γ_ε (Eq. 1.3) is an example of a one-parameter Lie group. The following is another example of a Lie group that is a symmetry for the equation (Eq. 1.1):

$$P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y}) = (x, e^\lambda y)$$

This is a symmetry because it maps a point on one solution curve of the equation (Eq. 1.1) to a point on another solution curve of the equation (Eq. 1.1).

2 Symmetry Condition

The **total derivative operator** is important for understanding the symmetry condition. The total derivative operator is:

$$D_x = \delta_x + y'\delta_y + y''\delta_{y'} + \cdots \qquad \text{(Eq. 2.1)}$$

In general, we are working with differential equations of the form:

$$\frac{dy}{dx} = \omega(x, y) \qquad \text{(Eq. 2.2)}$$

In order to satisfy the symmetry condition, the point $(\hat{x}, \hat{y})$ must also be on a solution curve to the differential equation (Eq. 2.2),

$$\therefore \frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}) \qquad \text{(Eq. 2.3)}$$

Written with the derivative operator, the equation (Eq. 2.3) is,

$$\frac{d\hat{y}}{d\hat{x}} = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{\hat{y}_x + \frac{dy}{dx}\hat{y}_y}{\hat{x}_x + \frac{dy}{dx}\hat{x}_y} = \omega(\hat{x}, \hat{y})$$

From equation (Eq. 2.3) we know that $\frac{dy}{dx} = \omega(x, y)$,

$$\therefore \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}) \qquad \text{(Eq. 2.4)}$$

The equation (Eq. 2.4) can be used in a few cases to determine the symmetries of a differential equation.

For example, consider

$$\frac{dy}{dx} = y \qquad \text{(Eq. 2.5)}$$

The symmetry of this equation should satisfy the equation (Eq. 2.4). In fact, it should satisfy,

$$\frac{\hat{y}_x + y\hat{y}_y}{\hat{x}_x + y\hat{x}_y} = \hat{y} \qquad \text{(Eq. 2.6)}$$

We need to solve the equation for $\hat{x}$ and $\hat{y}$ to obtain the symmetry of the differential equation.

Let us assume that $\hat{x} = \hat{x}(x, y)$ and $\hat{y} = y$.

Hence, the equation (Eq. 2.6) becomes,

$$\frac{y}{\hat{x}_x + y\hat{x}_y} = y \implies y = y(\hat{x}_x + y\hat{x}_y)$$

As long as $y \neq 0$,

$$1 = \hat{x}_x + y\hat{x}_y \tag{Eq. 2.7}$$

Since $\hat{x}_x = 1$ and $\hat{x}_y = 0$, the following symmetry will satisfy the equation (Eq. 2.4)

$$(\hat{x}, \hat{y}) = (x + \lambda, y),\ \lambda \in \mathbb{R}$$

2.1 Orbits

Orbits are an essential tool for solving differential equations using symmetry methods. Suppose, there is a point A on a solution curve to a differential equation. Under a given symmetry, the orbit of A is the set of all points that A can be mapped to for all possible values of λ.

3 New Coordinate system

As $\lambda \in \mathbb{R}$ varies under a given symmetry $P_\lambda : (x, y) \mapsto (\hat{x}, \hat{y})$, a point A travels along its orbit. The tangent vectors to an orbit under a given symmetry are crucial to determining the new coordinate system $(r(x, y), s(x, y))$.

3.1 Tangent Vectors

Tangent vector to orbits at the point $(\hat{x}, \hat{y})$ are described by the tangent vector in the x direction, denoted by $\xi(\hat{x}, \hat{y})$ and the tangent vector in the y direction, denoted by $\eta(\hat{x}, \hat{y})$. Thus,

$$\frac{d\hat{x}}{d\lambda} = \xi(\hat{x}, \hat{y});\ \frac{d\hat{y}}{d\lambda} = \eta(\hat{x}, \hat{y})$$

$$\left(\left.\frac{d\hat{x}}{d\lambda}\right|_{\lambda=0}, \left.\frac{d\hat{y}}{d\lambda}\right|_{\lambda=0} \right) = (\xi(x, y), \eta(x, y))$$

The tangent vectors are useful for finding invariant solution curves[1]. In an invariant solution curve the derivative at the point (x, y) will point in the direction of the tangent vectors to the orbit. As λ varies, the point is mapped to another point on the same solution curve.

$$\therefore \frac{dy}{dx} = \omega(x, y) = \frac{\eta(x, y)}{\xi(x, y)}$$

[1] An invariant solution curve is always mapped to itself under a symmetry. The points on an invariant solution curve are mapped either to themselves or to another point on the same curve

Peter E. Hydon defines *characteristic* Q as,

$$Q(x, y, y') = \eta(x, y) - y'\xi(x, y) \qquad \text{(Eq. 3.1)}$$

Since, $\frac{dy}{dx} = \omega(x, y)$, we can rewrite the equation (Eq 3.1) as the *reduced characteristic* $\bar{Q}$ as,

$$\bar{Q}(x, y, y') = \eta(x, y) - \omega(x, y)\xi(x, y)$$

A solution curve $y = f(x)$ is invariant if and only if $Q(x, y, y') = 0$ when $y = f(x)$.

The Lie symmetries are trivial if and only if $Q(x, y, y')$ is identically zero, that is, $\eta(x, y) \equiv \omega(x, y)\xi(x, y)$.

3.2 Canonial Coordinates

If a differential equation has a symmetry of the form $(\hat{x}, \hat{y}) = (x, y + \lambda)$, then it can be reduced to quadrature and can be solved by an integrating technique. However, not all differential equations have a symmetry of this form in Cartesian coordinates. Therefore, one can change to a new coordinate system in $(r(x, y), s(x, y))$ to obtain a symmetry: $P_\lambda : (r, s) \mapsto (\hat{r}, \hat{s}) = (r, s + \lambda)$. The tangent vectors at (r, s) when $\lambda = 0$ are

$$\left.\frac{d\hat{r}}{d\lambda}\right|_{\lambda=0} = 0; \quad \left.\frac{d\hat{s}}{d\lambda}\right|_{\lambda=0} = 1 \qquad \text{(Eq. 3.2)}$$

Applying the chain rule to the equation (Eq. 3.2) we get,

$$\left.\frac{d\hat{r}}{d\lambda}\right|_{\lambda=0} = \left.\frac{d\hat{r}}{dx}\frac{dx}{d\lambda}\right|_{\lambda=0} + \left.\frac{d\hat{r}}{dy}\frac{dy}{d\lambda}\right|_{\lambda=0} = \frac{dr}{dx}\xi(x, y) + \frac{dr}{dy}\eta(x, y) = r_x\xi(x, y) + r_y\eta(x, y) = 0 \qquad \text{(Eq. 3.3)}$$

and

$$\left.\frac{d\hat{s}}{d\lambda}\right|_{\lambda=0} = \left.\frac{d\hat{s}}{dx}\frac{dx}{d\lambda}\right|_{\lambda=0} + \left.\frac{d\hat{s}}{dy}\frac{dy}{d\lambda}\right|_{\lambda=0} = \frac{ds}{dx}\xi(x, y) + \frac{ds}{dy}\eta(x, y) = s_x\xi(x, y) + s_y\eta(x, y) = 1 \qquad \text{(Eq. 3.4)}$$

The change of coordinates should be invertible in some neighbourhood of (x, y), so we impose the nondegeneracy condition,

$$r_x s_y - r_y s_x \neq 0 \qquad \text{(Eq. 3.5)}$$

This condition ensures that if a curve of constant s and a curve of constant r meet at a point, they cross one another transversely. Any pair of functions $(r(x, y), s(x, y))$ satisfying the equations (Eq. 3.3), (Eq. 3.4) and (Eq. 3.5) is called a pair of

canonical coordinates. Using the method of characteristics, we can write *symmetric equations* for the equation (Eq. 3.3) as,

$$\frac{dx}{dt} = \xi,\ \frac{dy}{dt} = \eta,\ \frac{dr}{dt} = 0 \implies \frac{dx}{\xi} = \frac{dy}{\eta} \qquad \text{(Eq. 3.6)}$$

Similarly, we can write *symmetric equations* for the equation (Eq. 3.4) as,

$$\frac{dx}{dt} = \xi,\ \frac{dy}{dt} = \eta,\ \frac{ds}{dt} = 1 \implies \frac{dx}{\xi} = \frac{dy}{\eta} = ds \qquad \text{(Eq. 3.7)}$$

Consider, the function $\phi(x, y)$, the first integral of a differential equation:

$$\frac{dy}{dx} = f(x, y) \qquad \text{(Eq. 3.8)}$$

Therefore, $\phi(x, y) = c$, where c is a constant.[2]. Applying first derivative operator to $\phi(x, y)$, we get,

$$\phi_x + f(x, y)\phi_y = 0,\ \phi_y \neq 0 \qquad \text{(Eq. 3.9)}$$

If we divide the equation (Eq. 3.3) by $\xi(x, y)$, we get

$$r_x\frac{\xi(x, y)}{\xi(x, y)} + r_y\frac{\eta(x, y)}{\xi(x, y)} = 0,\ \xi(x, y) \neq 0 \implies r_x + r_y\frac{\eta(x, y)}{\xi(x, y)} = 0$$

Comparing the result with equation (Eq 3.9), we find that $r(x, y)$ is a first integral of

$$\frac{dy}{dx} = \frac{\eta(x, y)}{\xi(x, y)},\ \xi(x, y) \neq 0$$

Therefore, $r(x, y) = c$, where c is a constant. To find s we use equation (Eq. 3.7):

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} = ds \implies s = \int \frac{dx}{\xi(x, y)} = \int \frac{dy}{\eta(x, y)}$$

If $\xi(x, y) = 0$ then, $r(x, y) = x$ and $s = \int \frac{dy}{\eta(x,y)}$

4 Solving ODEs with Lie Symmetries

We need to write the differential equation in terms of r and s in order to solve it. Then, put the solution back into Cartesian coordinates. To find $\frac{ds}{dr}$, apply the total derivative operator to get

$$\frac{ds}{dr} = \frac{s_x + \omega(x, y)s_y}{r_x + \omega(x, y)r_y} \qquad \text{(Eq. 4.1)}$$

[2]First integrals are nonconstant functions whose value is constant along solution curves of the equation (Eq. 3.8)

This will result in an equation $\frac{ds}{dr}$ written in terms of x and y. To write it in terms of r and s, solve the coordinates $r(x, y)$ and $s(x, y)$ for x and y, then simplify. From there, solve the equation and put the solution back into Cartesian coordinates.

4.1 The Linearized Symmetry Condition

In practice, it is difficult to find a symmetry that works for a given differential equation. In order to find the symmetry it is necessary to solve the symmetry condition given in equation (Eq. 2.4)

$$\frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y})$$

This equation gives the symmetry $(x, y) \mapsto (\hat{x}, \hat{y})$. In general, this is a complicated nonlinear partial differential equation in the two unknowns x and y.

Therefore, it is necessary to use a *linearized symmetry condition* to find $\xi(x, y)$ and $\eta(x, y)$.

We can linearize the symmetry condition using a Taylor series expansion. We can expand $\hat{x}$, $\hat{y}$ and $\omega(\hat{x}, \hat{y})$ around $\lambda = 0$,

$$\hat{x} = x + \lambda\xi(x, y) + O(\lambda^2) \qquad \text{(Eq. 4.2)}$$

$$\hat{y} = y + \lambda\eta(x, y) + O(\lambda^2) \qquad \text{(Eq. 4.3)}$$

Here, $O(\lambda^2)$ denotes the error function in the Taylor series expansion of $\hat{x}$,$\hat{y}$ and $\omega(\hat{x}, \hat{y})$. To simplify notation, the arguments (x, y) will be omitted from ξ, η and ω from now on.

On substituting the equations (Eq. 4.2) and (Eq. 4.3) in symmetry condition (Eq. 2.4), we get:

$$\frac{\omega + \lambda(\eta_x + \omega\eta_y) + O(\lambda^2)}{1 + \lambda(\xi_x + \omega\xi_y) + O(\lambda^2)} = \omega(x + \lambda\xi + O(\lambda^2), y + \lambda\eta + O(\lambda^2)) \qquad \text{(Eq. 4.4)}$$

We now expand each side of equation (Eq 4.4) as a Taylor series about $\lambda = 0$, assuming that each series converges.

$$\omega + \lambda\{\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2\} + O(\lambda^2) = \omega + \lambda\{\xi\omega_x + \eta\omega_y\} + O(\lambda^2) \qquad \text{(Eq. 4.5)}$$

This condition is necessarily satisfied at $\lambda = 0$, which corresponds to the trivial symmetery $(\hat{x}, \hat{y}) = (x, y)$. Equating the $O(\lambda^2)$ terms gives the **linearized symmetry condition**,

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2 = \xi\omega_x + \eta\omega_y \qquad \text{(Eq. 4.6)}$$

The linearized symmetry condition can be rewritten in terms of the reduced characteristic, $\bar{Q} = \eta - \omega\xi$ as follows,

$$\bar{Q}_x + \omega\bar{Q}_y = \omega_y\bar{Q} \qquad \text{(Eq. 4.7)}$$

Each solution of the equation (Eq. 4.7) corresponds to infinitely many Lie groups, for if $\bar{Q}$ satisfies the equation (Eq. 4.7) then $(\xi, \eta) = (\eta, \bar{Q}+\omega\xi)$ is a tangent vector field of a one-parameter group, for any function ξ .All trivial Lie symmetries correspond to the solution $\bar{Q} = 0$ of the equation (Eq. 4.7). In principle, the nontrivial symmetries can be found from the equation (Eq. 4.7) by using the method of characteristics.

5 Example

$$\frac{dy}{dx} = \frac{y}{x} + x \qquad \text{(Eq. 5.1)}$$

Solution:

Substituing the equation (Eq. 5.1) into the linearised symmetry condition, we get

$$\eta_x + (\eta_y - \xi_x)\left(\frac{y}{x} + x\right) - \xi_y\left(\frac{y}{x} + x\right)^2 - \xi\left(1 - \frac{y}{x^2}\right) - \eta\left(\frac{1}{x}\right) = 0$$

In its current form, solving the equation is a very difficult task. Therefore, we can make an ansatz about ξ and η. Suppose that $\xi = 0$ and η is a function of x only. Then we get,

$$\eta_x - \frac{\eta}{x} = 0 \implies \int \frac{d\eta}{\eta} = \int \frac{dx}{x} \implies \ln\eta = \ln x + c_0 \implies \eta = cx$$

Now we can find the canonical coordinates r and s. Recall from Section 3.2 that when $\xi(x, y) = 0$, $r = x$. To find s we can solve,

$$ds = \frac{dy}{\eta}$$

Therefore,

$$s = \int \frac{dy}{\eta} = \int \frac{dy}{cx} = \frac{y}{cx}$$

Now set $c = 1$ to get,

$$(r(x, y), s(x, y)) = \left(x, \frac{y}{x}\right)$$

and

$$s_x = -\frac{y}{x^2}\ ,\ s_y = \frac{1}{x}$$

Now we can substitute r and s into the equation (Eq. 4.1) to obtain

$$\frac{ds}{dr} = \frac{-\frac{y}{x^2} + \frac{1}{x}\left(\frac{y}{x} + x\right)}{1} = 1$$

Therefore, $s = r + k$, where k is a constant. Substituting x and y back in, we get

$$\frac{y}{x} = x + k$$

and general solution of equation (Eq. 5.1) is $y = x^2 + kx$.

6 Conclusion

This article describes the method to solve first order ordinary differential equations using Lie Symmetry groups. These methods are extended to solve higher order ordinary differential equations. It is often quite easy to find symmetries of a given differential equation (even an unfamiliar one) and to use them systematically to obtain exact solutions. Symmetries can also be used to simplify problems and to understand bifurcations of nonlinear systems.

References

[1] Hydon, P.E., *Symmetry Methods for Differential Equations; A Beginner's Guide*, Cambridge Texts in Applied Mathematics, Cambridge University Press (2000)

[2] Steinhour, R.A. (2013), *The Truth About Lie Symmetries: Solving Differential Equations With Symmetry Methods*, The College of Wooster Libraries; Open Works, Senior Independent Study Theses. Paper 949, 1-51

[3] Starrett, J. (Nov., 2007),*Solving Differential Equations by Symmetry Groups*, The American Mathematical Monthly, 114 (9): 778-792

KUSHAGRI TANDON, B.Sc.(H) MATHEMATICS, 4th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI
kushagritandon28@gmail.com

Interdisciplinary Aspects of Mathematics

Mathematics is just not a classroom discipline but a tool for organizing and understanding various concepts and applications. This section covers topics that delve into other disciplines, integrating the mode of thinking and knowledge of the respective discipline with Mathematics. The section hence highlights the cosmic scope of Mathematics, leveraging its amalgamation with other disciplines.

Capital Asset Pricing Model

Namrata Lathi

Abstract

This paper discusses about Capital Asset Pricing Model, a tool used in finance to determine a theoretically required rate of return of an asset by taking into account the asset's sensitivity to non-diversifiable risk. This paper also demonstrates the derivation of the CAPM Model.

1 About the Model

A fundamental question in finance is how the risk of an investment should affect its expected return. The **Capital Asset Pricing Model** provided the first coherent framework for answering this question. It was developed in the early 1960s by William Sharpe (1964), Jack Treynor (1962), John Lintner (1965) and Jan Mossin (1966) independently.

The Capital Asset Pricing Model describes the relationship between **systematic risk** and **expected return for assets**, particularly stocks. It explains variations in the rate of return on a security as a function of the rate of return on a market portfolio.

The model says that the investors need to be compensated in two ways: **time value of money** and **risk**. The **time value of money** is represented by the **risk-free rate** i.e. the iterest rate on government securities like Treasury bills. The risk is calculated by taking a risk measure-**beta**.Beta reflects **how risky an asset** is compared to overall market risk.

2 Some basic concepts related to the model

1. Investors face two kinds of risks: systematic risk and unsystematic risk
2. **Systematic Risk**- It is the risk of recession, enactment of unfavorable regulation, etc.
3. **Unsystematic Risk** - It is the risk specific to a particular investment. For example, major disruption in the companys supply chain
4. Diversification of portfolio reduces variability. This is because prices of different stocks generally don't move together. An increase in one is offset by the decrease in another.

5. Expected return on a market portfolio = weighted average of the expected returns on the individual stocks

6. **Variability** of portfolio is determined using **variance** and **standard deviation**

3 Assumptions of the Model

All investors-

- are wealth maximizers
- can borrow or lend unlimited amounts at a risk free rate
- have the same expectations from the market
- are price takers i.e. they cannot influence prices
- trade without transaction or taxation costs
- deal with securities that are fully divisible
- assume all information is available at the same time to all investors

4 The CAPM Model

$$r_a = r_f + \beta_a(r_m - r_f)$$

$$\begin{aligned} \text{where } r_f &= \text{Risk free rate} \\ \beta_a &= \text{Beta of the security} \\ r_m &= \text{Expected market return} \\ (r_m - r_f) &= \text{Equity market premium} \end{aligned}$$

5 Calculation of Beta (β)

Beta is the measure of sensitivity of change in price of a security in relation to changes in the overall stock market. It is a **function** of the **volatility of the asset** and the **market** as well as the **correlation** between the two.
It is calculated as follows-

$$\beta = \frac{\sigma_{im}}{\sigma_m^2}$$

where σ_{im} is the covariance between the stock returns and market returns and σ_m^2 is the variance of the returns on the market

6 Derivation of the Model

Efficient Frontier
It is the locus of all the efficient portfolios i.e. the locus of all those portfolios which offer the highest expected return for any level of risk. In other words, we can also say that it is the locus of all those feasible portfolios which have the smallest variance for a prescribed expected return.
Suppose-

- there are m risky securities
- expected return on ith security = μ_i
- covariance of return between the i^{th} and j^{th} security = σ_{ij}
- variance of return on the i^{th} security = $\sigma_{ii}= \sigma_i^2$, $\sigma_i^2 > 0$
- δ_i = percentage of the value of a portfolio invested in the ith security, hence $\sum_1^m \delta_i = 1$
- no security can be represented as a linear combination of the other securities, i.e. the variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$, is non-singular

Thus, the frontier can be described as the set of portfolios which satisfy the constrained minimization problem,

$$\text{Minimize Z} = \frac{1}{2}\sigma^2 \tag{1}$$

$$\text{s.t. } \sigma^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \delta_i \delta_j \sigma_{ij} \tag{2}$$

$$\mu = \sum_{i=1}^{m} \delta_i \mu_i \tag{3}$$

$$\sum_1^m \delta_i = 1 \tag{4}$$

Using Lagrangian multipliers λ_1 and λ_2 we can rewrite above equation as min $\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\delta_i\delta_j\sigma_{ij} = \lambda_1(\mu - \sum_{i=1}^{m}\delta_i\mu_i) + \lambda_2(1 - \sum_{1}^{m}\delta_i)$

A critical point occurs where the partial derivatives of above equation w.r.t. $\delta_1, ..., \delta_m, \lambda_1$ and λ_2 are equal to 0 i.e.

$$
\begin{aligned}
0 &= \sum_{1}^{m}\sigma_j\sigma_{ij} - \lambda_1\mu_i - \lambda_2, i = 1, ..., m && (5)\\
0 &= \mu - \sigma_1^m\delta_i\mu_i && (6)\\
0 &= 1 - \sum_{1}^{m}\delta_i && (7)
\end{aligned}
$$

Further, the δ's which satisfy the above set of equations minimize σ^2 and are unique on the assumption on Ω.

As the above set of equations $5, 6, 7$ are linear in the $\delta's$ and hence we have from equation 5 that

$$\delta_k = \lambda_1\sum_{1}^{m}V_{kj}\mu_j + \lambda_2\sum_{1}^{m}V_{kj}, k = 1, 2, ..., m \tag{8}$$

where the V_{ij} are defined as the elements of the inverse of the variance-covariance matrix, i.e. $\Omega^{-1} = [V_{ij}]$
Multiplying equation 8 by μ_k and summing over $k = 1, 2, ..., m$ we have that

$$\sum_{1}^{m}\delta_k\mu_k = \lambda_1\sum_{1}^{m}\sum_{1}^{m}V_{kj}\mu_j\mu_k + \lambda_2\sum_{1}^{m}\sum_{1}^{m}V_{kj}\mu_k, k = 1, 2, ..., m \tag{9}$$

and by summing equation 8 over $k = 1, 2, ..., m$, we have that

$$\sum_{1}^{m}\delta_k = \lambda_1\sum_{1}^{m}\sum_{1}^{m}V_{kj}\mu_j + \lambda_2\sum_{1}^{m}\sum_{1}^{m}V_{kj} \tag{10}$$

Define
A=$\sum_{j=1}^{m}\sum_{k=1}^{m}V_{kj}\mu_j$
B=$\sum_{j=1}^{m}\sum_{k=1}^{m}V_{kj}\mu_j\mu_k$
C=$\sum_{j=1}^{m}\sum_{k=1}^{m}V_{kj}$

From equations 6,7,9 and 10, we have a simple linear system of λ_1 and λ_2

$$\mu = \text{B}\lambda_1 + \text{A}\lambda_2 \quad (11)$$
$$1 = \text{A}\lambda_1 + \text{C}\lambda_2 \quad (12)$$

where due to the symmetry of the variance-covariance matrix we find that

$$\sum_{j=1}^{m} \sum_{k=1}^{m} V_{kj}\mu_k = \sum_{j=1}^{m} \sum_{k=1}^{m} V_{kj}\mu_j$$

and that $B > 0, C < 0 \, \text{and} \, D > 0$

Solving eqn 11 and 12 for λ_1 and λ_2, we get

$$\lambda_1 = \frac{C\mu - A}{D} \quad (13)$$
$$\lambda_2 = \frac{B - A\mu}{D} \quad (14)$$

where $D = BC - A^2 > 0$

By solving the optimization problem we can thus find the efficient frontier in the $\sigma^2 - \mu$ space:

$$\sigma_p^2 = \frac{C\mu_p^2 - 2A\mu_p + B}{D}$$

This implies

$$\sigma_p = \sqrt{\frac{C\mu_p^2 - 2A\mu_p + B}{D}}$$

This is the equation of a **hyperbola**.

Also

$$\frac{d^2\sigma}{d\mu^2} = \frac{1}{D\sigma^3} > 0$$

This implies σ is strictly convex function of μ.
So, the graph of efficient frontier is as follows-

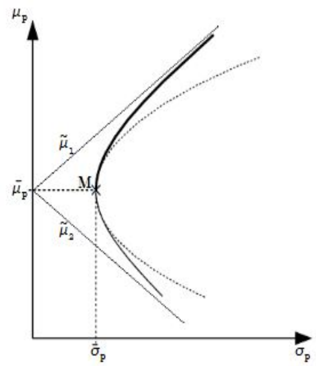


Figure 1: The efficient Frontier in the σ-μ space is the upper part of a hyperbola with the asymptotes μ_1 and μ_2

Now as per assumption 2, the investors can borrow and lend money at the risk-free interest rate r_f.
This means if the investor invests some money in T-Bills and place the remainder in common stock portfolio T, she can obtain any combination expected return and risk along the straight line joining r_f and T as shown in the following figure:

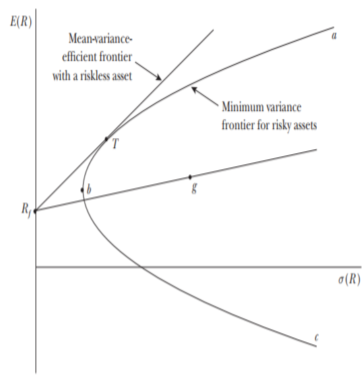


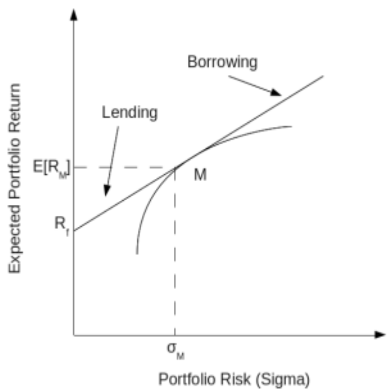
Figure 2: Investment opportunities

This is because:
The return, expected return and standard deviation of return on portfolios of the risk-free asset f and a risky portfolio g vary with x, the proportion of portfolio funds invested in f, as

$$\begin{aligned} R_p &= xR_f + (1-x)R_g \\ E(R_p) &= xR_f + (1-x)E(R_g) \\ \sigma(R_p) &= (1-x)\sigma(R_g), x < 1 \end{aligned}$$

which together imply that the portfolios plot along the line from R_f through T **The best portfolio is the one that lies at the tangency point of the straight line to the efficient frontier.**
The tangent line from the intercept point on the efficient frontier to the place where the expected return on a holding equals the R_f is called **Capital market Line**.

Now, the relationship between risk and expected return from T-Bills and the market portfolio can be plotted as follows-



.

So, for any portfolio lying on the line, expected return R_p will be equal to:

$$R_p = R_f + \frac{R_m - R_f}{\sigma_m}\sigma_p$$

where R_p is the return on an efficient portfolio
R_f is the risk-free rate
R_m is the return on the market portfolio
σ_m is the standard deviation of returns on the market portfolio
σ_p is the standard deviation of returns on efficient portfolio p

Now, as CML is a straight line,

$\Rightarrow$ efficient portfolios that plot along CML are perfectly correlated i.e. $\rho_{pm} = 1$, hence the above equation can be written as -

$$R_p = R_f + \frac{(R_m - R_f)\rho_{pm}}{\sigma_m}\sigma_p$$

where: ρ_{pm} is the correlation coefficient between the return of portfolio p and that of the market.

$$\begin{aligned} R_p &= R_f + \frac{\rho_{pm}\sigma_p}{\sigma_m}(R_m - R_f) \\ &= R_f + \frac{\rho_{pm}\sigma_p\,\sigma_m}{\sigma_m^2}(R_m - R_f) \\ &= R_f + \frac{Cov(R_p.R_m)}{\sigma_m^2}(R_m - R_f) \\ R_p &= R_f + \beta_p(R_m - R_f) \end{aligned}$$

This is the CAPM Model.

7 Conclusion

The version of the CAPM developed by Sharpe (1964) and Lintner (1965) has never been an empirical success. It suffered various criticisms because of its assumptions like assuming β as a constant, use of historical data as input, assuming all investors have same information, assuming variance of returns as an adequate measurement of risk and a few more.
Despite this, the CAPM is a very well established model for practical problems. It is favored for its simplicity. It serves as an effective tool for teaching the fundamental concepts of portfolio theory and asset pricing.

References

[1] Brealey,R., Myers,S. and Allen,F., *Principles of Corporate Finance*-10th *Edition*, McGraw Hill Companies

[2] Merton,R.(1970),*An Analytic Derivation of the Effective Portfolio Frontier*,Alfred P. Sloan School of Management,493-70

[3] Deeley, C.(2012),*A simple derivation of the CAPM model from CML*,Charles Sturt University

[4] Kure, T.(2010),*Geometric Analysis of the Capital Asset Pricing Model*,Norwegian School of Economics and Business Administration

[5] Fama,F.,French,K.(2010),*The Capital Asset pricing Model: Theory and Evidence*,Graduate School of Business, University of Chicago and Tuck School of Business, Dartmouth College,18:25-46

NAMRATA LATHI, B.Sc.(H) MATHEMATICS, 4th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI
namratalathi@gmail.com

Latin Squares: Recreation, Disproving Euler to Cryptography

Jaya Sharma

Abstract

This paper discusses about the concept of Latin Squares. It gives an account of their history and mathematical developments. Their practical applications in technology, statistical analysis and puzzles have also been explained.

1 Euler, Orthogonality and Latin Squares

Late in his life, Euler wrote a long paper on a simple puzzle:

Consider a group of 36 *officers of* 6 *different ranks, taken from* 6 *different regiments. Your task is to arrange them in a square such that in* ***each row and colum****,* *there are* ***6 officers of each rank and regiment.***

We define a "**Latin square**"as an $n \times n$ matrix of n symbols (or **order** $\boldsymbol{n}$), such that each symbol occurs ***exactly once*** in every row and column. Each distinct rank, and similarly each regiment, is denoted by a natural number from ***1 to n***.

Consider two Latin squares A and B (separately arranged for ranks and regiments). Let a_{ij} *and* b_{ij} denote the entries in the i^{th} row and the j^{th} column of A and B respectively. Then A and B are said to be "**mutually orthogonal Latin squaress**"($MOLS$) if the n^2 ordered pairs (a_{ij}, b_{ij}) are distinct and all $2n$ entries (n ranks and n regiments) are found in each row and column. The square obtained by superposing 2 $MOLS$ (as shown below) shall be be called a "**Graeco-Latin Square**".

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} (3,3) & (1,2) & (2,1) \\ (2,2) & (3,1) & (1,3) \\ (1,1) & (2,3) & (3,2) \end{bmatrix}$$

Fig 1: Obtaining Graeco-Latin Square of order 3

Hence the solution of Euler's problem was to find a **pair *MOLS* of order *6***. Euler could find $MOLS$ when $n = 3, 4, 5$, odd or divisible by 4, but not for $n = 2, 6$. He went on to conjecture that *no solutions existed for orders* $n = 4k + 2$, $k \epsilon N$. In fact it wasn't until 1900 that the conjecture was proven for $n = 6$.

$$\begin{bmatrix} (1,2) & (2,1) \\ (2,1) & (1,2) \end{bmatrix}, \begin{bmatrix} (1,1) & (2,4) & (3,2) & (4,5) & (5,3) \\ (2,2) & (3,5) & (4,3) & (5,1) & (1,4) \\ (3,3) & (4,1) & (5,4) & (1,2) & (2,5) \\ (4,4) & (5,2) & (1,5) & (2,3) & (3,1) \\ (5,5) & (1,3) & (2,1) & (3,4) & (4,2) \end{bmatrix}$$

Fig 2: Only possible case when $n = 2$, Graeco-Latin Square for $n = 5$

2 Further Developments and Disproving Euler

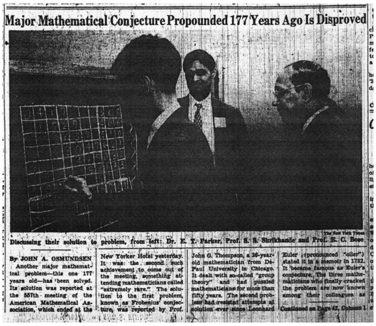
2.1 Mac Neish's Conjecture

Let $n(k)$ and $N(k)$ denote the minimum and maximum number of *MOLS* of order k, respectively. H. F. MacNeish [1922] showed that the value ***n(k)*** would be ***one less than the smallest prime number that appears when k is written out as a product of its prime factors***. Later MacNeish conjectured that ***N(k)*** $=$ ***n(k)***. Thus, if true, MacNeish's conjecture would imply Euler's conjecture to be correct.
However in 1958, E. T. Parker disproved the conjecture by showing that there exist at least ***three*** *MOLS* of order 21 (there should be at most 2 according to MacNeish's conjecture). This result although didn't disprove Euler's conjecture, threw serious doubts on it.

2.2 Final Solution

The impetus of Parker's work led Raj C. Bose, who was Professor of Statistics at the University of North Carolina, to develop some powerful general methods (by using concepts like finite groups and fields, difference sets, orthogonal arrays and tools in experimental design, especially the "incomplete block designs") for construction of higher order Graeco- Latin Squares. Bose and S. S. Shrikhande (his student) succeeded in applying these methods to the construction of a Graeco-Latin Square of ***order 22***. *Euler's conjecture was at last proven* ***false***.

By refining results and further use of other designs in statistically controlled experiments, it was later proven false for all values of $n = 4k + 2 > 6$. Parker also



Major Mathematical Conjecture Propounded 177 Years Ago Is Disproved

The New York Times

Discussing their solution to problem, from left: Dr. E. T. Parker, Prof. S. S. Shrikhande and Prof. R. C. Bose

By JOHN A. OSMUNDSEN

Another major mathematical problem—this one 177 years old—has been solved. Its solution was reported at the 557th meeting of the American Mathematical Association, which ended at the New Yorker Hotel yesterday. It was the second such achievement to come out of the meeting, something attending mathematicians called "extremely rare." The solution to the first problem, known as Frobenius' conjecture, was reported by Prof. John G. Thompson, a 26-year-old mathematician from De-Paul University in Chicago. It dealt with so-called "group theory" and had puzzled mathematicians for more than fifty years. The second problem had resisted attempts at solution ever since Leonhard Euler (pronounced "oiler") stated it in a memoir in 1782. It became famous as Euler's conjecture. The three mathematicians who finally cracked the problem are now known among their colleagues as

Continued on Page 42, Column 1

Fig 3: As Reported on front page of NYT

46	57	68	70	81	02	13	24	35	99
71	94	37	65	12	40	29	06	88	53
93	26	54	01	38	19	85	77	60	42
15	43	80	27	09	74	66	58	92	31
32	78	16	89	63	55	47	91	04	20
67	05	79	52	44	36	90	83	21	18
84	69	41	33	25	98	72	10	56	07
59	30	22	14	97	61	08	45	73	86
28	11	03	96	50	87	34	62	49	75
00	82	95	48	76	23	51	39	17	64

Fig 4: A Graeco-Latin square of order 10

developed a new method of constructing an infinite series of Graeco-Latin Squares of orders $10, 14$, etc.

3 Applications of Latin Squares

3.1 Agronomic And Medical Research

MOLS are useful in the design of experiments wherever there are ***many variables involved***. Consider testing the effects of n different agricultural fertilisers on the growth of n different crops. So, we can divide the field into cells of a $n \times n$ square and apply the fertilisers and plant crops in the pattern of *MOLS* of order n. Square hence obtained would show the combinations of crop and fertiliser applied in each cell. Because of this pattern, one can use statistical analysis to eliminate the bias due to variation in natural soil fertility along a row or column of the field. They are also widely used in medical research and biological research. For example, one may need to test a medical treatment for its efficacy on patients of different age, weight, stages of a disease etc. A similar statistical experiment method called "***Balanced Incomplete Block Designs*** "is also used.

3.2 Error Correcting Codes

Orthogonal Latin squares are used as error correcting codes in situations where communication is disturbed by noise. A letter in the message to be sent is encoded by sending a series of signals at different frequencies at successive time intervals. In the example below, the letters A to H are encoded by sending signals at four different frequencies, in four time slots. The letter C, for instance, is encoded by first sending at frequency **3**, then **4**, **1** and **2**. The ***encodings*** of the eight letters are formed from ***MOLS***.

$$\begin{bmatrix} (A) & 1 & 2 & 3 & 4 \\ (B) & 2 & 1 & 4 & 3 \\ (C) & 3 & 4 & 1 & 2 \\ (D) & 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} (E) & 1 & 3 & 4 & 2 \\ (F) & 2 & 4 & 3 & 1 \\ (G) & 3 & 1 & 2 & 4 \\ (H) & 4 & 2 & 1 & 3 \end{bmatrix}$$

Fig 5: Encoding of letters at different frequencies

Now imagine that there's added noise in channels **1** and **2** during the whole transmission. The letter A would then be picked up as: **12 12 123 124**. Because of the noise, we can no longer tell if the first two slots were **1,1** or **1,2** or **2,1** or **2,2** . But inferring from the table of encodings, **1,2** case is the only one that yields a matching letter in the above table. Hence the letter A has been identified and recovered. Similarly, we may imagine a burst of static over all frequencies in the third slot for A:**1 2 1234 4**. Again we can easily infer the letter being sent. The number of ***errors*** this code can spot is ***one less than the number of time slots***. It has also been proven that if the *number of frequencies is a prime or a power of a prime*, *MOLS* produce error detecting codes that are most efficient.

3.3 Cryptography

Latin Squares are used as codes. Two *MOLS* are taken and letters or symbols are assigned to each of the co-ordinates. Then the sender encrypts the message by ***substituting each letter/symbol with the corresponding co-ordinates***.The receiver then decrypts the message by using the *MOLS*, since every co-ordinate occurs *only once*. Unless a third party knows the two *MOLS* used and which letters correspond to which co-ordinates, its *impossible to decrypt*, making it an effective coding system. For example, consider the Graeco-Latin Square and the letter matrix:

$$\begin{bmatrix} (1,3) & (2,1) & (3,2) \\ (2,2) & (3,3) & (1,1) \\ (3,1) & (1,2) & (2,3) \end{bmatrix} \begin{bmatrix} D & C & I \\ B & H & A \\ G & F & E \end{bmatrix}$$

Fig 6: Encoding using a Graeco-Latin Square

Then to send the word 'BAG' one could just send the string of co-ordinates **(2, 2)(1, 1)(3, 1)**.

With larger *MOLS*, more letters and symbols could be encrypted. There is also more secrecy because the ***number of pairs of MOLS increases exponentially as the value of n gets higher***. Latin Squares are also used in ciphers and cryptographic functions.

3.4 Puzzles And Games

Any *solution* to the popular *Sudoku* puzzle is a ***Latin square***, with the additional restriction that nine particular adjacent subsquares must also contain the digits 1–9. The similar *Ken-Ken* puzzles are also examples of Latin squares where the objective is to fill the grid in with the digits 1 through 6 like a sudoku puzzle and each bold-outlined group of cells should contain digits which achieve the specified result using the specified mathematical operations. Latin squares have been used as the basis for several board games, notably the popular abstract strategy game *Kamisado*.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5

Fig 7: Sudoku

11+ 5	2÷ 6	3	20× 4	6× 1	2
6	3- 1	4	5	3÷ 2	3
240× 4	5	6× 2	3	6	1
3	4	6× 1	7+ 2	30× 5	6
6× 2	3	6	1	4	9+ 5
8+ 1	2	5	2÷ 6	3	4

Fig 8: Ken-Ken

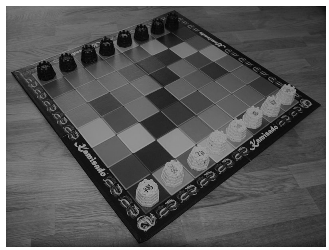


Fig 9: Kamisado

References

[1] Frisinger,H.H.(1981), *The Solution Of A Famous Two-Centuries-Old Problem: The Leonhard Euler-Latin Square Conjecture*, Historia Mathematica, (8): 56-60.

[2] Tang,J.(2009), *Latin Squares and Their Applications*, University of Queensland.

[3] en.wikipedia.org/wiki/Latin_square

[4] thewire.in/194729/ss-shrikhande-math-latin-square-euler/

[5] mathworld.wolfram.com/EulersGraeco-RomanSquaresConjecture.html

[6] www.cecm.sfu.ca/organics/papers/lam/paper/html/NYTimes.html

Further Reading

[1] Bose,R.C., Shrikhande,S. S. and Parker,E. T.(1960), *Further Results On The Construction Of Mutually Orthogonal Latin Squares And The Falsity Of Euler's Conjecture*, Canadian Journal of Mathematics, (12): 189-203.

JAYA SHARMA, B.Sc.(H) MATHEMATICS, 2nd SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI
forjayasharma@gmail.com

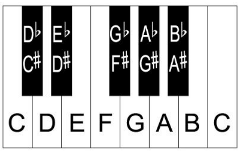
Delving Relationship Between Mathematics And Music

Akanksha Gupta

Abstract

Music is an alliance between arts and science and Mathematics plays a central role in this alliance. This paper discusses the study of these two disciplines, mainly focused on the mathematical part that exist in music through examples from the compostion of French Composer, Olivier Messiean.

1. Fundamentals Of Music



This is the set of 12 keys of a piano keyboard which repeats itself to form the entire keyboard. When you move from one key to the just next key, say from F to $F^{\sharp}$, it is called a *semitone*. Two semitones form a whole tone, for example: from F to G is a whole tone. $\sharp$ and $\flat$ denote accidentals. $\sharp$ is called *sharp* i.e, higher in pitch by a semitone and $\flat$ is called *flat* means lower in pitch by a semitone.

Table 1

Number of Semitones Apart	**Music Interval**	**Notes**	**Ratio**
1	Minor 2nd	C-C$\sharp$	16:15
2	Major 2nd	C-D	9:8
3	Minor 3rd	C- D$\sharp$	6:5
4	Major 3rd	C-E	5:4
5	Perfect 4th	C-F	4:3
6	Augmented 4th/Diminished 5th	C- F$\sharp$	7:5
7	Perfect 5th	C- G	3:2
8	Minor 6th	C-G$\sharp$	8:5
9	Major 6th	C-A	5:3
10	Minor 7th	C-A$\sharp$	7:4
11	Major 7th	C-B	15:8
12	Octave	C-C	2:1

A **musical interval** is the difference between two pitches [1]. The smallest difference is the semitone. In physical terms, it is the ratio of the frequency of two tones.

[1] how low or high a note sounds

Scale is a set of notes in order of their pitch. If the pitch is becoming higher, it is called an ascending scale. If the pitch is becoming lower, it is called a descending scale.

2. Historical Connections Between Mathematics And Music

Pythagoras, besides being a mathematician, was also a music theorist. We owe him for the discovery of the fundamental correspondence between musical intervals and numerical ratios. Each musical interval is associated to the ratio of frequency of the higher note to that of the lower-pitched note.

The most basic musical interval is the prime,where the fundamental[2] note is played in comparison to itself. The ratio of frequency is $1:1$. The second most basic interval, is the octave, where the fundamental relates to a second note that has double the frequency of the fundamental.The ratio of the fundamental and the second note is $1:2$.

After the prime interval, the octave is the second most pleasant sounding interval. The interval of fifth corresponds to the numerical ratio $3:2$. Once Pythagoras established the ratio of the octave and the fifth, he used these relationships and simple mathematics to obtain further intervals.

Illustration 1: *The Second(Major)*

The interval C-G is the fifth, with C being the fundamental. When G is the fundamental, the interval G-D is the fifth. By a factor of 3/2, D is higher than G and G is higher than the original fundamental C. Thus, to compare the frequencies of C and D, all that is required is multiplication.

$\therefore$ the frequency ratio of C-D

$= (3:2) \times (3:2)$

$= 9:4$

D must now be transposed down one octave. Recall that the frequency ratio of the note one octave below the fundamental and the fundamental itself is the ratio $1:2$. Multiplying again gives the required ratio.

$\therefore$ the frequency ratio of C-D

$(9:4) \times (1:2)$

$= 9:8$

$\Rightarrow$ any interval with the ratio $(9:8)$ is a second(major).

Table 2

[2] root/ initial note

Note	C	D	E	F	G	A	B	C
Freq. Ratio	1	9/8	32/27	4/3	3/2	27/16	16/9	2

Illustration 2: *The Minor Second*

The smallest difference between whole tones exists between the notes E-F and B-C. The interval D-E is a second, so multiplying the frequency of E by 8 : 9 gives the frequency of D. the interval D-F is a third, so multiplying the frequency of D by 32 : 27 gives the frequency of F.

$\therefore$ the frequency ratio E-F

$= (8 : 9) \times (32/27)$

$= 256 : 243$

The inversion of the interval B-D(a sixth) multiplied by the seventh D-C gives the frequency ratio for B-C.

$\therefore$ the frequency ratio B-C

$= (16 : 27) \times (16 : 9)$

$= 256 : 243$

In both of these cases, the resulting frequency ratio is 256 : 243, and is known either as the semitone step, or the minor second.

The Pythagorean scale has many beautiful properties. Fourth and fifth, the building blocks of all other intervals, are all pure intervals.This scale of one example from Ancient Greece. Composition and creation of scales has differed but it has been considered the building blocks of musical composition. Since the 18th century, there has been general acceptance of the *tempered*[3] *scale*. Each unit in a tempered scale is a tempered semitone, with a value of $\sqrt[12]{2}$

The eight intervals of the octave have frequency ratios:

$$1, f, f^2, f^3, f^4, f^5, f^6, f^7, f^8, f^9, f^{10}, f^{11}, f^{12}$$

where $f^{12} = 2(\Rightarrow f = \sqrt[12]{2})$ The interval, in semitones, between any two tones of the tempered scale is: $12 \times \log_2$ (frequency ratio).

From Table 1, we find that most of the ratios of music intervals are of the form $(n+1)/n$, where n is a positive integer. This arithmetic form is called superparticular ratios. Following are few questions concerning superparticular ratios mathematically as well as music oriented.

1. Can the square root of a superparticular ratio be superparticular? Or at least be rational?

[3]**Tempered Scale:** divides an octave into twelve equal parts, e.g: piano keyboard(set of 12 keys).

Musically related question is: Can we divide a consonant interval[4] into two equal consonant intervals?

2. Given a superparticular ratio, can we enumerate all the various ways of expressing it as a product of superparticular ratios? Is this number finite or infinite? Is the number of possibilites finite if we fix a bound on the number of factors?

   This question is related to the question of diving a consonant interval into certain number of consonant intervals and the problem of constructing scales whose all intervals are consonant.

The response to Question 1 is known since antiquity. A proof of the fact that there is no rational fraction whose square is equal to a superparticulr ratio. A more general result is: for any pairs of integers B, C whose quotient is equal to a superparticular ratio, there is no sequence of integers $D, E, F, ...N$ between B and C satisfying $B/D = D/E = E/F = \ldots = N/C$.

In Boethius' *Musical Instituion* III.5 d[4], the author mentions that in order to circumvent the impossibility of dividing the tone (9/8) into two equal parts.

Regarding Question 2, we note that any superparticular ratio can be written as a product of two or more, using the following:

$$\frac{p+1}{p} = \frac{n(p+1)}{np} = \frac{np+n}{np+n-1} \times \frac{np+n-1}{np+n-2} \times \frac{np+n-2}{np+n-3} \times \cdots$$

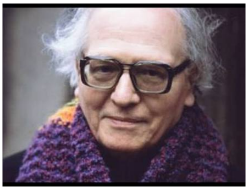
In particular, above relation shows that every superparticular ratio can be written as a product of superparticular ratios in an infinite number of ways. Aristides Quintilianus, in Book III of his *De Musica*[5], used this method to describe the division of the tone as $17/16 \times 18/17 = 9/8$, then the following division of semitones and of quarter tones : $33/32 \times 34/33 = 17/16$ and $35/34 \times 36/35 = 18/17$

The concept of Golden Ratio also exists in music. The Golden Section is often used to generate rhythmic change or to develop a melody line, and is found in the musical timing of compositions. The climax of a song, for example, is often found at the point of the golden ratio(approximately 61.8% of the way through a composition). This is often also the place where significant changes in key or chord structures are placed.

Note: In the next section there has been use of integers mod 12 for the twelve notes of the octave.

[4] pleasant intervals

3. Messiaen: The Mathematics of his Musical Language



Olivier Messiaen(1908-1992), a French composer and organist, was a great contributor to contemporary music and thinking. Even though he never considered himself as a mathematician, he granted mathematics to prominent place, both in his composition and in his theortical teaching. He was fascinated with time and rhythm and it is his contributions regarding time and rhythm made his work so unique. He uses the term "charm of impossibilites" to explain how in his musical language, "certain mathematical impossibilities, certain closed circuits, possesses a strength of bewitchment, a charm". His three principle innovations, *Mode of Limited Transposition(MOLT)*, *non-retrogradable rhythm* and *symmetric permutations* describe this power.

3.1. Modes of Limited Transposition

Mathematically, a mode[5] of limited transposition[6] is a sequence of distinct notes that has a nontrivial group of symmetries by transpositions mod 12.

C	C♯/D♭	D	D ♯/E♭	E	F	F♯/G♭	G	G♯/A♭	A	A♯/B♭	B
0	1	2	3	4	5	6	7	8	9	10	11

Using the fact that one octave is made up of twelve semitones, and the number twelve is divisible by various numbers, Messiean formed the modes by dividing the octave into different recurring groups, each being a tiny transposition, each group has an identical order of intervals, and the last pitch of one group serves as the first pitch of the next.

The original form of each mode is called the first transposition, and always begins on the note C. each transposition thereafter, begins on subsequent chromatic steps. Each group within mode is constructed in the same way, so only a limited transposition would result in new mode. Thus he created the term mode of limited transposition.

- **Mode 1-** This is the sequence of notes C, D, E, F♯, G♯, B♭. He means this mode as being "two times transposable" and by this means that the corresponding pitch class(pc)-set[7] is invariant by the translation $x \to x + 2$.

[5]a sequence of distinct notes which describe the atmosphere(colour) of a musical composition

[6]writing music in a different key, mathematically, let $T_n : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ be such that $T_n(x) = x + n$ mod 12

[7]set of all pitches that are whole number of octaves apart. eg: *pc set* of "C" - $\{C_n : n \in \mathbf{Z}\}=\{\ldots, C_{-1}, C_0, C_1, \ldots\}$

The *pc*-set [0, 2, 4, 6, 8, 10] and it is the orbit[8] of 0.This scale is called whole tone scale because each degree increases by a tone. It corresponds to the division of octave into six equal subintervals.

- **Mode 2-** This mode is "three times transposable", which means that, as a subset of $\mathbb{Z}_{12}$, it is invariant under the translation $x \to x+3$. It is given by the sequence of notes C, D♭, E♭, E♮, F♯, G, A, B♭, which corresponds to the *pc-set* [0, 1, 3, 4, 6, 7, 9, 10],i.e, union of the orbits of 0 and 1 by maps $x \to x+3$. It consists of 4 groups of 2 consecutive elements in $\mathbb{Z}_{12}$.

- **Mode 3-** This mode is "four times transposable", and it is defines by the sequence of notes C, D, E♭, E♮, F♯, G, A♭, B♭, B♮. In $\mathbb{Z}_{12}$, i.e, the sequence of integers mod 12 [0, 2, 3, 4, 6, 7, 8, 10, 11], which is the union of the orbits of 0, 2 and 3 by the map $x \to x+4$. The *pc-set* consists in 3 groups of 3 consecutive elements in $\mathbb{Z}_{12}$.

- **Mode 4-** This is the sequence C, D♭, D♮, F, F♯, G, A♭, B, which corresponds to the *pc-set* [0, 1, 2, 5, 6, 7, 8, 11]. It is the union of the orbits of 0, 1, 2 and 5 by the map $x \to x+6$.

- **Mode 5-**This is the sequence C, D♭, F, F♯, G, B, which corresponds to the *pc-set* [0, 1, 5, 6, 7, 11]. It is the union of the orbits of 0,1 and 5 by the map $x \to x+6$.

- **Mode 6-** This is the sequence of C, D, E, F, F♯, G♯, A♯, B which corresponds to the *pc-set* [0, 2, 4, 5, 6, 8, 10, 11]. It is the union of the orbits of 0, 2, 4 and 5 by the map $x \to x+6$.

- **Mode 7-** This is the sequence C, D♭, D♮, E♭, E♮, F♯, G, A♭, A♮, B♭, which corresponds to the *pc-set* [0, 1, 2, 3, 4, 6, 7, 8, 9, 10], It is the union of the orbits of 0,1,2,3 and 4 by the map $x \to x+6$.

3.2. Non-retrogradable Rhythm

Messiean's second creator of the "charm of impossibilites" is non-retrogradable rhythms. He discovered these rhthymic patterns in Indian and Greek rhythms. The existence of a rhythmic palindrome, i.e, a special rhythmic form that is the same whether it is read backwards or forwards. He thus named them as "non-retrogradable rhythm".

[8]corresponds to modes which are at their first transposition

For simple rhythms which have only three values such as 2, 1, 2, rhythmic pattern holds if the outer two values are identical, and surround what he called a "free central value". Here 1 is free central value. When rhythm are more complex and contain more than three values, he extended his principle: "all rhythms divisible into two groups, one of which is the retrograde of the other, with a common central value, are non-retrogradable". These rhythms are simple mathematical patterns, yet he believed they held philosphic importance. According to him, strengths of these patterns are:

1. Since patterns are pallindromic in nature, this creates an irreversibility of time, whether time moves forward or backwards, the events are the same.
2. These rhythms have powerful link to our temporal life. The two outer groups in his analogy are the past and the future. the middle, free central value is the present. he claims that we cannot distinguish between past and future without the freedom of the present. Hence, it is not possible to distingush outer two groups without the central value in the rhythm.

3.3 Symmetrical Permutations

This is his third *mathematical impossibilty.* He calls, symmetrical permutations as permuatations which have a small group of symmtries. To obtain such permutations, one start at the center of the sequence of notes, and then takes successively one note from the right and one note from the left, until one reaches the two ends of the sequence, and of its iterates. For instance, this process transforms the sequence 1, 2, 3, into the sequence 2, 1, 3. Applying the same rule to 2, 1, 3, we obtain 1, 2, 3, which is the sequence we started with. Thus, the permutation 1, 2, 3 $\rightarrow$ 2,1,3 is of order of 2. Let us consider now a sequence of 4 notes. The iterates are: $1, 2, 3, 4 \rightarrow 2, 3, 1, 4 \rightarrow 3, 1, 2, 4 \rightarrow 1, 2, 3, 4$. This time order of permutation is 3.

4. The Music of Olivier Messiean

These three innovations gave his music a charm, a sense of bewitchment. Each innovation formed a complete group, and a closed circuit which would always go back to the beginning. They gave him way of describing his religious belief: with Catholic faith, you will always return to the truth of eternity. He used the mathematical techiniques of his musical language, to transcend the temporal limitations of music, and express his faith.

5. Conclusion

We found that relation between mathematics and music is immense. Many mathematicians throughout history have been interested in music and considered themselves as music theorist. On the other hand, Messiean never considered himself as a mathematician but gave mathematics a prominent place in his compositions to express his ideas and beliefs. **Is further analysis needed?** Mathematicians contributed to the wealth of knowledge on music theory, often by writing books and sharing their ideas. Messiean used his interest and knowledge for his composition. Did the published and shared work of mathematicians influence musicians? To enhance the arguement extra research is needed. But indeed, numbers play a crucial role in expressing colours of our feelings whether musicians know it or not.

References

[1] Papadopoulos A.(2014), *Mathematics and Group Theory in Music*, Cornell University Library, 1-28

[2] Shah, S.(2010), *An Exploration of the Relationship between Mathematics and Music*, Manchester Institute for Mathematical Sciences, The University of Manchester, 5-96

[3] Townsend A.(2011),*Maths and Music Theory*, Talk in University College London Undergraduate Maths Coloquium, 5

[4] Boethius(1873), ed. by Gottfried *De institutione musica(Musical institution)*

[5] Quintilianus A. commentary by Mathiesen T. J.(1983) *De Musica(On Music)*, Yale University Press

Further Readings

[6] Loy G.(2006), *Musimathics: Mathematical Foundation of Music*

[7] Wright D.(2009), *Mathematics and Music*, 28 of Mathematical World

AKANKSHA GUPTA, B.Sc.(H) MATHEMATICS, 4th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN, NEW DELHI

akan1298gupta@gmail.com

Mathematics in Solar Energy

Vanshika and Shubhra Aggarwal

Abstract

This paper aims to cover all the topics that are relevant for getting a broad overview on the use of mathematics in Solar Energy, with a focus on the following topics: Calculation of the Annual Solar Energy Output of a Photo Voltaic System, Geometry, Shape, Positioning, Characterisation, Cost Effectiveness and Area of Solar Panels along with an example.

1 Introduction

"How can it be that mathematics, being after all a product of human thought
which is independent of experience,
is so admirably appropriate to the objects of reality?"
-Albert Einstein

Solar energy works by capturing the energy of the sun and turning it into electricity .Sun is a natural nuclear reactor. It releases tiny packets of energy called photons, which travel the 93 million miles from the sun to Earth in about 8.5 minutes. Every hour, enough photons impact our planet to generate enough solar energy to theoretically satisfy global energy needs for an entire year.
When photons hit a solar cell, they knock electrons loose from their atoms. If conductors are attached to the positive and negative sides of a cell, it forms an electrical circuit. When electrons flow through such a circuit, they generate electricity. Multiple cells make up a solar panel, and multiple panels (modules) can be wired together to form a solar array.

2 Shape of Solar Panels

Solar Panels can be found in every shape be it Rhombus, Parallelogram, Kite, Square, Triangle etc.Researchers at Massachusetts Institute of Technology(MIT) recently discovered that creating a 3D-inspired solar panel not only helps to keep up with the trends, it could draw in 20 times more energy than flat panel designs. Traditional solar panels lay flat on a surface or rooftop, facing the sun to collect energy. MIT researchers decided to change the shape of solar panels, conducting experiments with a cube, tall cube, and tower-shaped panels to see which design brought in more energy. Compared to flat panels, all three 3D panels created impressive results and outproduced traditional panels, with the accordion-style tower drawing in 20 times more power per square foot. The accordion-style arrays also work better because they receive solar energy from all angles rather than in just one direction.

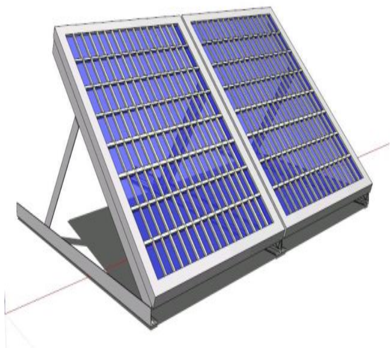


Figure 1: 3d Solar Panel

The MIT study states that making these improvements can help power output become "more predictable and uniform, which could make integration with the power grid easier than with conventional systems."

2.1 Geometry of Solar Panels

Researchers at Northwestern University have now developed a new design for organic solar cells that could lead to more efficient, less expensive solar power. Instead of attempting to increase efficiency by altering the thickness of the solar cell's polymer layera tactic that has preciously garnered mixed resultsthe researchers sought to design the geometric pattern of the scattering layer to maximize the amount of time light remained trapped within the cell.
Using a mathematical search algorithm based on natural evolution, the researchers pinpointed a specific geometrical pattern that is optimal for capturing and holding light in thin-cell organic solar cells.
The resulting design exhibited a three-fold increase over the Yablonovitch Limit, a thermodynamic limit developed in the 1980s that statistically describes how long a photon can be trapped in a semiconductor.
In the newly designed organic solar cell, light first enters a 100-nanometer-thick "scattering layer", a geometrically-patterned dielectric layer designed to maximize the amount of light transmitted into the cell. The light is then transmitted to the

active layer, where it is converted into electricity.
"We wanted to determine the geometry for the scattering layer that would give us optimal performance," said Cheng Sun, assistant professor of mechanical engineering in Northwestern's McCormick School of Engineering and Applied Science and co-author of the paper."But with so many possibilities, it's difficult to know where to start, so we looked to laws of natural selection to guide us."[10]
The researchers employed a genetic algorithm, a search process that mimics the process of natural evolution, explained Wei Chen, Wilson-Cook Professor in Engineering Design and professor of mechanical engineering at McCormick and co-investigator of the research.
"Due to the highly nonlinear and irregular behavior of the system, you must use an intelligent approach to find the optimal solution," said Cheng. "Our approach is based on the biologically evolutionary process of survival of the fittest."
The researchers began with dozens of random design elements, then "mated" and analyzed their offspring to determine their particular light-trapping performance. This process was carried out over more than 20 generations and also accounted for evolutionary principles of crossover and genetic mutation.
The resulting pattern will be fabricated with partners at Argonne National Laboratory.

2.2 Positioning of Solar Panels

To make the predictions about the best way to fix solar panels where you live,use Google earth or maps.Find your house and then zoom in.

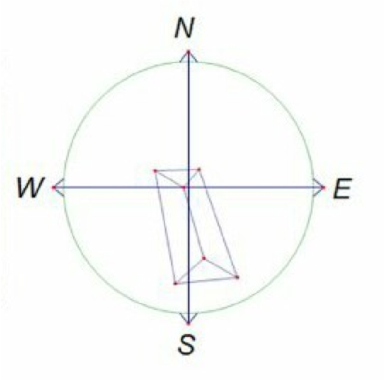


Figure 2

Make a quick sketch of the orientation of your house and then suggest which part of the roof would be best to put the panels on.
To decide which part of the roof,angle of inclination helps. Angle of inclination can

be 0°,10°,20°,30°and so on. The best angle can be in between any of the above two angles as well.
Consider the following steps to get an idea about the positioning of the solar panel:

- Place the Bearing alignment circle on top of a book or a folder so that it rests horizontally.
- Place the compass on top of the alignment Circle.
- Turn the alignment circle until north on the paper is the same as the needle on the compass.

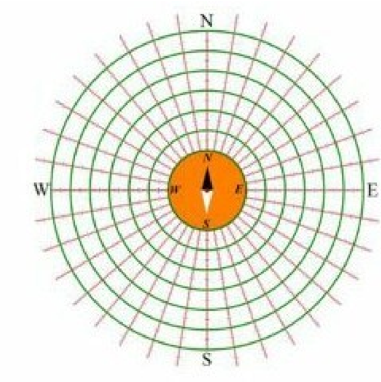


Figure 3

- Now take the compass away and place the ruler in the middle of the circle standing on one end.
- Turn the ruler abouty its vertical axis until the edges of the shadow are at right angles to the ruler.Through this the bearing of the sun is lined up and it can be read by looking at the angle value directly opposite the shadow.
- Now the solar cell is ready to be lined up with the sun.
- Connect the solar panel with the multimeter as can be seen in the photograph.Connect the multimeter to one cell only.
- Take your solar panel,ruler and alignment circle and set them as in the diagram.

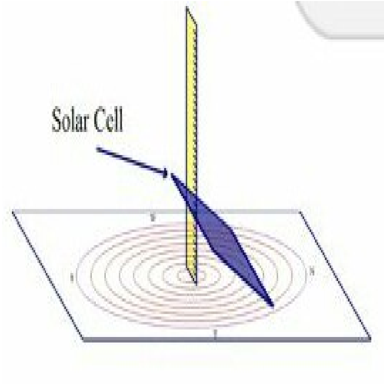


Figure 4

- Lean the solar panel against the ruler and change the angle until the voltage reaches a maximum.Note the value on the multimeter.The angle which has been found out is called the pitch angle.
- Calculating Pitch Angle- Measure the height of the solar cell from the ruler.Remember to allow for the bottom part of the ruler where there are no markings.
- Measure the horizontal distance from the base of the ruler to the solar cell.
- Now we use trigonometry to calculate the Pitch angle. See the diagram to calculate the pitch angle.

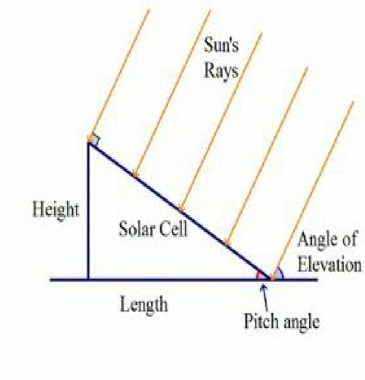


Figure 5

- The angle of elevation of sun = 90°-pitch angle.

Circuit Setup:
The solar panel should be connected to the multimeter as shown in Figure 6. Start with the dial on the multimeter in the 'OFF' position. The black plug connects the

central COMM socket on the multimeter to the black(negative) socket on any one of the four cells (labelled 1 to 4). In this picture, it is connected to cell number 1. The red plug connects the right-hand socket on the multimeter to the red (positive) socket of the same cell.

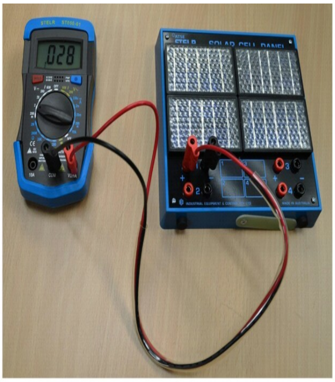


Figure 6

Global solar radiation on a tilted surface I_T consists of daily direct solar radiation I_b , diffuse solar radiation I_d , and ground reflected radiation I_r. Daily solar radiation on a tilted surface for a given month can be estimated as follows:

$$I_T = I_b + I_d + I_r$$

The daily direct radiation on a tilted surface I_b can be obtained by means of R_b , the ratio of the average daily direct radiation on a tilted plane to that on a horizontal plane and the parameters to it correlated:

$$\begin{aligned} I_b &= \frac{\cos\theta}{\cos\theta_z} = H_b R_b, \\ R_b &= \cos\beta \\ &\quad - \sin\beta(\sin\delta\cos\phi\cos\gamma \\ &\quad - \cos\delta\sin\phi\cos\omega\cos\gamma - \cos\delta\sin\gamma\sin\omega) \\ &\quad (\sin\delta\sin\phi + \cos\delta\cos\phi\cos\omega)^{-1}, \end{aligned}$$

where β is the slope of the panel as to the horizontal plane,γ is the azimuth, δ is the solar declination, ω is the solar hour angle, ϕ is the latitude, and θ and θ_z are the solar incidence angle on the considered plane and the solar zenith angle, respectively. If we consider a uniform and isotropic distribution of diffuse solar radiation over the sky hemisphere, I_d would be easily obtained from the simple approximation:

$$I_d(iso) = \frac{H_d(1+\cos\beta)}{2}$$

where H_d is the horizontal sky diffuse irradiation.
The evaluation of the ground-reflected diffuse radiation depends on I_r. Most studies consider that the ground reflection process is ideally isotropic, in which a specific case I_r can be simplified as follows:

$$I_r = \frac{H_g \rho (1 - \cos\beta)}{2}$$

where ρ represents the diffuse reflectance of the ground (also called ground albedo) and H_g is the horizontal global solar irradiation.
Finally, the daily global solar irradiation on slopes I_T can be expressed as the sum of I_b,I_r and $I_d(iso)$.

3 Energy produced by Solar Panel

The total amount of energy they produce during the day is governed by things like solar latitude which is associated with latitude and season, and atmospheric conditions such as cloud coverage, temperature and degree of pollution apart from panel orientation and shading. In India, ideal orientation for solar panels is slight tilt towards true south.
Power under the curve:

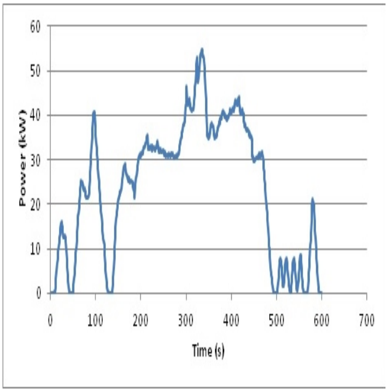


Figure 7

Solar panels often include measuring devices which record the amount of power that is being generated every few minutes.
For constant power, if we plot the power (kW) on the vertical axis and the time (hours) on the horizontal axis, then the energy can be interpreted as the area under the curve :

$$\begin{aligned} \text{Energy} &= \text{power} * \text{time} \\ &= \text{height(kW)} * \text{width(hrs)} \\ &= \text{area under the power curve in units of kWh.} \end{aligned}$$

Even when the power curve is not constant, the energy produced can still be interpreted as the area under the curve. To find the energy generated by the solar panels, area can be approximated using geometric shapes such as rectangles and triangles As the number of rectangles becomes larger (i.e. goes to infinity) and their width goes to zero, the area under the rectangles approaches the area under the curve. This limiting process is the base for the definition of the integral as the limit of Riemann sums.

3.1 Space required to install 1kW Solar Panels

Under clear skies and good sunshine, each square meter is receiving about 1000 watts of solar energy. At typical 15 % panel efficiency, a 1 sq m area will generate 150 watts of power. For 1 kW power output, about 7 sq m area will be required. After leaving some free space, about 10-12 sq m clear roof area will be required.

4 Calculation of the Annual Solar Energy Output of a Photo Voltaic System

The global formula to estimate the electricity generated in output of a photo voltaic system is :

$$E = A * r * H * PR$$

where,

$$
\begin{aligned}
\text{E} &= \text{Energy(kWh)} \\
\text{A} &= \text{Total solar panel Area (m2)} \\
\text{r} &= \text{solar panel yield(\%)} \\
\text{H} &= \text{Annual average solar radiation on tilted panels} \\
\text{PR} &= \text{Performance ratio, coefficient for losses} \\
& \quad \text{(range between 0.5 and 0.9, default value } = 0.75)
\end{aligned}
$$

5 Area of Solar Panels

Solar panels are usually installed at an angle to the earth's surface. A solar panel has a conversion efficiency of 100 percent,i.e, it converts all the solar energy into electrical energy.
Area of solar panels depends on its efficiency.Higher the efficiency,lower is the area required.
Example:
Suppose that 20 solar panels rated at 100 watts each and having a conversion efficiency of 18 percent need to be installed.
The total power output of the solar system can be calculated as :
Total Power Output = Total Area*Solar Irradiance*Conversion Efficiency
Required Total Output Power = 20 panels*100 watts =2000 watts
The solar irradiance for a surface perpendicular to the Sun's rays at sea level on a clear day is about $1000\,\text{watt/m}^2$
Also the conversion efficiency is 18 percent.
Putting these values we get: 2000 watts = Total area*$(1000\,\text{watt/m}^2) * 0.18$ which implies that total area = 11.11m^2.

6 Characterization of Solar Panels

Solar panels are characterized by number of watts (Wp) they can produce under Standard Test Conditions (STC) of 1000W/m^2 irradiation, cell temperature of 25 ° Celsius and air mass of 1.5. This is their peak performance. . The number 1.5 has been agreed upon for the STC (Standard Test Condition) for testing solar panels. Air mass is a measure of the distance traveled by sunlight through the Earths atmosphere.

7 Cost Effectiveness

To determine whether installing solar panels are worth it might require knowing the cost to buy and install the solar panels, the cost for upkeep (minimal) , the price of electricity in the region and the reduced carbon footprint that would arise from using solar panels. The reduction in greenhouse gas output could be of benefit to the environment and society. Solar technology is improving and the cost of going solar is dropping rapidly, so our ability to harness the suns abundance of energy is on the rise.

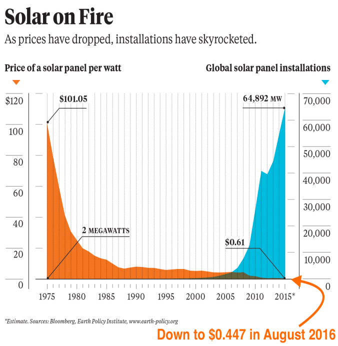


Figure 8

It can be seen that there is a negative correlation between the prices and installations.

8 Conclusion

Unglazed transpired collectors or UTC (also known as perforated collectors) are a relatively new development in solar collector technology, introduced in the early nineties for ventilation air heating. These collectors are used in several large buildings in Canada, USA and Europe, effecting considerable savings in energy and heating costs.

A 2017 report from the International Energy Agency shows that solar has become the worlds fastest-growing source of power marking the first time that solar energys growth has surpassed that of all other fuels. In the coming years, we will all be enjoying the benefits of solar-generated electricity in one way or another.

References

[1] Tyagi, H. *Applications of Solar Energy*, Springer, 2017

[2] Calabro,E.(2013), *An Algorithm to Determine the Optimum Tilt Angle of a Solar Panel from Global Horizontal Solar Radiation*, Journal of Renewable Energy,(2013): 12 pages

[3] https://livingonsolarpower.wordpress.co.in/2013/03/23/basic-solar-energy-math/amp/

[4] www.mathaware.org

[5] http://www.motherearthnews.com/renewable-energy/solar-power-systems-zmaz98onzraw

[6] http://photovoltaic-software.com/PV-solar-energy-calculation.php

[7] www.raymaps.com

[8] www.solarmango.com

[9] www.slideshare.net

[10] http://www.mccormick.northwestern.edu/news/articles/2013/01/evolution-inspires-more-efficient-solar-cell-design.html

VANSHIKA JAIN, B.Sc.(H) MATHEMATICS, 4th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN,NEW DELHI
vanshikaj48@gmail.com
SHUBHRA AGGARWAL, B.Sc.(H) MATHEMATICS, 6th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN,NEW DELHI
aggarwal.shubhra@yahoo.co.in