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This jounal is dedicated to Maryam Mirzakhani, the first woman to be awarded the Fields Medal.

PREFACE

This edition of Éclat, 2014 - 2015 has been a remarkable journey taking us through twelve brilliant papers with incessant rounds of editing to bring out the final version. It has been a great challenge to give a platform to this journal which was only an idea a few years ago. With this copy of Eclat, we aim to provide readers with an in-depth knowledge into various topics of mathematics. It has been a pleasure to put together research and insights into black and white.

The following compilation contains four categories of mathematics namely, History of mathematics, Rigour in mathematics, Inter disciplinary aspects of mathematics and Extension of course content.We have hoped to furnish all varities of readers with suitable content in each field. This journal will not just help you grow in your mathematical understanding but also takes you down to a thought process of coming up with your own ideas.

The entire department of Mathematics of our college has been instrumental in the publication of this journal.We sincerely thank the faculty of the Department of Mathematics, Lady Shri Ram College For Women, for guiding and supporting us throughout the year. We welcome corrections, suggestions and submissions from our readers.

We dedicate this volume of the journal to Ms. Maryam Mirzakhani, the first woman to win the Fields medal, one of the highest honours in Mathematics. The award recognizes her sophisticated and monumental contributions to the fields of geometry and dynamical systems, particularly in understanding the symmetry of curved surfaces. She is an inspiration to all female mathematicians.
Happy Exploring!

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CONTENTS

History of Mathematics

Mathematics is the one of the oldest academic discipline involving stimulating and intriguing concepts. It is far beyond the ken of one individual and to make any contribution to the evolution of ideas, an understanding of the motivation behind the ideas is needed. The section covers the genesis of mathematical ideas, the stream of thought that created the problem and what led to its solution. The aim is to acquaint the readers with historically important mathematical vignettes and make them inured in some important ideas of Mathematics.

LEONHARD EULER

RABIKA GURUNG AND SHIREEN PRAKASH

Abstract. A significant figure in the history of Mathematics, Euler's momentous contributions to varied fields of mathematics like analysis and group theory account for almost one third of the total mathematical works done during his lifetime. It is a virtually impossible task to do justice, in a short span of time and space, to the great genius of Leonhard Euler.

Personal life

Leonhard Euler was born in Basel, Switzerland on April 15, 1707 to Paul Euler and Marguerite Brucker. Euler studied theology and Hebrew in the University of Basel. He took the freshman courses on elementary mathematics given by Johann Bernoulli. Euler pursued his mathematical studies with such zeal that he soon caught the attention of Bernoulli, who encouraged him to study more advanced books on his own and even offered him assistance at his house every Saturday afternoon. These personal meetings, which have also been mentioned by Euler in his autobiography of 1767, have become famously known as the *privatissima*, and they continued well beyond his graduation. Euler recounts this early learning experience at the university in his brief autobiography of 1767:

"In 1720, I was admitted to the university as a public student, where I soon found the opportunity to become acquainted with the famous professor Johann Bernoulli, who made it a special pleasure for himself to help me along in the mathematical sciences. Private lessons, however, he categorically ruled out because of his busy schedule. However, he gave me a far more beneficial advice, which consisted in myself getting a hold of some of the more difficult mathematical books and working through them with great diligence, and should I encounter some objections or difficulties, he offered me free access to him every Saturday afternoon, and he was gracious enough to comment on the collected difficulties, which was done with such a desired advantage that, when he resolved one of my objections, ten others at once disappeared, which certainly is the best method of making happy progress in the mathematical sciences."

In 1723, Euler graduated with a master's degree and delivered a public lecture (in Latin) comparing Descartes' system of natural philosophy with that of Newton. Euler's years at the Academy of St. Petersburg from 1727 to 1741 proved to be a period of extraordinary productivity and creativity. In 1733, Danielle Bernoulli returned to free Switzerland and hence Euler, at the early age of 26, stepped into the leading mathematical position in the

Academy. At St. Petersburg he married Catharina Gsell. Euler was very fond of children and had 13 of his own (out of which 5 died very young). Euler stayed in Russia till 1740 and after that settled in Berlin for the next 24 years of his life. Euler's eyesight worsened throughout his mathematical career. Three years after suffering a near-fatal fever in 1735, he became almost blind in his right eye. He later developed a cataract in his left eye, which rendered him almost totally blind. However, this had little effect on his work owing to his excellent mathematical skills and razor sharp memory. Euler's productivity in many areas of study actually increased. He produced, on an average, one mathematical paper every week in the year 1775. In St. Petersburg on 18 September 1783, after a lunch with his family, Euler suffered a brain haemorrhage. He died a few hours later.

Major works

- Euler's first independent work was done at the age of 19. He wrote paper on the masting of ships, proposed by the Paris Academy of Scieces. Though Euler failed to win the first prize, but he received an honourable mention and later he recouped the loss by winning the prize 12 times. Euler moved to St. Petersburg and then to Berlin. Moving to these two places gave a direction to his mathematical journey. In 1727, Euler received a call from St. Petersburg for the post of an associate of the medical section but owing to certain circumstances he ultimately joined where he belonged, that is the mathematical section.
- One of the most important works of this period was the treatise of 1736 on mechanics. His treatise for mechanics is equivalent to Descartes' work in geometry.
- In spite of the serious setbacks in health during his stay at St. Petersburg from 1727 to 1741, Euler astonishingly produced major works on mechanics, music theory, and naval architecture which are interspersed with some 70 memoirs on a great variety of topics that run from analysis and number theory to concrete problems in physics, mechanics, and astronomy.
- Euler left St. Petersburg on 19 June 1741 to take up a post at the Berlin Academy. In Berlin, he published the two works for which he would become most renowned: The *Introductio in analysin infinitorum*, a text on functions published in 1748, and the *Institutiones calculi differentialis*, published in 1755 on differential calculus. Due to his unpopularity in Frederick's court and his anger, Euler at the age of 59 again accepted the invitation of Catherine the Great to St. Petersburg. After Euler moved to St. Petersburg for the second time, his left eye also started worsening due to cataract. But he did not resign himself to darkness and silence. Before the last light faded, he accustomed himself to writing his formulae with a chalk on a large slate.The loss of his eyesight indeed sharpened his perceptions in the inner world of his imagination. One of the most remarkable features about Euler was his brilliant genius in both of the main currents or fields of mathematics.

Contributions

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- Mathematical Notation- Among his diverse works, the most notable was the introduction of the concept of functions. Euler also pioneered the use of '$f(x)$' to signify the function f applied to the argument x. He also defined the contemporary notation for the trigonometric functions, the letter 'e', for the base of the natural logarithm (Euler's number), the Greek letter '$\sum$ ' for summations and the letter 'i' to signify the imaginary unit.
- Analysis- Euler defined the use of the exponential functions and logarithms in analytic proofs. He discovered ways to state various logarithmic functions using power series, and he effectively defined logarithms for negatives and complex numbers. Through these accomplishments, he enlarged the scope of mathematical applications of logarithms to a great extent. Euler also explained in detail the theory of higher transcendental functions by inventing the gamma function and introduced a novel approach for solving quartic equations. He also discovered a technique to calculate integrals with complex limits, aiding the development of modern complex analysis and invented the calculus of variations along with the Euler Lagrange equation.
- Number Theory- Euler proved Fermat's little theorem, Newton's identities, Fermat's theorem on sums of two squares and he also distinctively contributed to Lagrange's four-square theorem. He significantly added value to the theory of perfect numbers, which had always been a captivating topic for several mathematicians.
- Physics and Astronomy- Euler made a noteworthy contribution in explaining the Euler Bernoulli beam equation, which became the foundation of engineering. He was awarded with numerous Paris Academy Prizes for his contributions in the field of astronomy. He found out the orbits of comets and other celestial bodies with exactness, understanding the nature of comets and calculating the parallax of the sun. This helped in preparing precise longitude tables.

Conclusion

Versatile mathematician Leonhard Euler made significant discoveries in varied fields which have been discussed above. All his collections, if printed, would occupy 60-80 quarto volumes reflecting his outstanding mathematical abilities. Euler is the only mathematician to have two numbers named after him: the important Euler's Number in calculus (e), approximately equal to 2.71828, and the Euler-Mascheroni Constant (gamma) also called "Euler's constant" approximately equal to 0.57721. He has been criticized, sometimes justly, for letting his mathematics run away with his sense of reality and that uncontrollable impulse to calculate merely for the sake of the beautiful analysis.

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GEORGE FRIEDRICH BERNHARD RIEMANN

SRISHTI BHOGAL AND SHIKHA KALIA

ABSTRACT. Sir George Friedrich Bernhard Riemann was an influential German mathematician who made lasting contributions to analysis, number theory, and differential geometry. He is the only one who established a geometric foundation for Complex Analysis through Riemann Surfaces through which multi-valued functions like the complex logarithm or the square root could become one-to-one functions.

INTRODUCTION

Bernhard Riemann *circa* 1863

George Friedrich Bernhard Riemann was born in Breselenz, a village near Dannenberg in the Kingdom of Hanover (in what is the Federal Republic of Germany today), on 7 September 1826. His father, Friedrich Bernhard Riemann, was a poor Lutheran pastor in Breselenz who fought in the Napoleonic Wars and his mother, Charlotte Ebell, died when he was just a child. Riemann was the second of six children; under-confident as a child with a fear of public speaking and susceptible to nervous breakdowns. On the other hand, he was a gifted mathematician with exceptional calculation aptitude. He was married to Elise Koch in 1862 and they had one daughter.

EARLY LIFE AND EDUCATION

In 1840 Bernhard Riemann went to middle school in Hanover (now Germany) and later attended high school at the Johanneum Lunerberg. In high school, Riemann studied the

Bible intensively, but he was often distracted by Mathematics. His teachers were amazed by his adept ability to perform complicated mathematical operations. In 1846, at the age of 20, he started studying Philosophy and Theology in order to become a pastor and help with his family's finances. During the spring of 1846,after gathering enough money, his father sent Riemann to the renowned University of Göttingen where he planned to study Theology but his focus soon shifted to Mathematics. Once there, he started studying Mathematics under Carl Friedrich Gauss (specifically his lectures on the method of least squares) who encouraged him to talk to his parents and switch to a degree in Mathematics. Riemann transferred to the University of Berlin in 1847 and remained there for the next two years. He returned to Göttingen in 1849.

Riemann became a lecturer on the recommendation of his teacher Gauss at the University of Göttingen, where he held his first lectures in 1854. In 1857, seeing the brilliance of Riemann, efforts were made to promote him to a position of an extraordinary Professor but this attempt failed and he was paid like any other professor in the University of Göttingen. In 1859, following Dirichlet's death, he was promoted to head the Mathematics Department at Göttingen.

Contributions to the Field of Mathematics

Bernhard Riemann had the touch of gold. Everything he worked with, he revolutionized. Riemann was a pure genius and his phenomenal contributions to the Mathematical world are a proof of his creativity and depth of knowledge. Despite his ailing health, he was one of the greatest mathematicians of all time. He had an extraordinary command over complex analysis which he interconnected with topology and number theory. Other revolutionary contributions include the tensor analysis, theory of functions, differential geometry and the most notable being the theory of manifolds. He pursued general actuality proofs, rather than constructive proofs that actually produce the objects. He said that this method led to theoretical clarity, making it easy for the mathematician and avoided getting confused with too much detail. The base of Einstein's Theory of Relativity was set up in 1854 when Riemann gave his first lecture.

Riemann Geometry

In 1854, Riemann presented his thoughts on geometry, for the post-doctoral qualification, to faculty member Gauss at Göttingen. Gauss was highly impressed with his ideas. Riemann argued that fundamental ingredients of geometry include space of points and it involves measuring distances along lines or curves in that space.

Riemann's idea was to introduce a collection of numbers at every point in space (i.e., a *tensor*) which would describe how much it was bent or curved. Riemann found that in four spatial dimensions, one needs a collection of ten numbers at each point to describe the properties of a $manifold$, no matter how distorted it is. This is the famous construction central to his geometry, known now as a *Riemannian metric*.

As per Riemann, the space does not need to be simple Euclidean space, but it could have many dimensions, even infinite dimensions. He also argued that it is not necessary that a surface be drawn completely in three-dimensional space. His work inspired Eugenio Beltrami, an Italian mathematician, to produce a description of non-Euclidean geometry. Albert Einsteins theory of relativity was based on Riemann's notions of geometry of space.

Complex Analysis

His contributions to this area are numerous. In his dissertation, he established a geometric foundation for complex analysis through $Riemann\ surfaces$, through which multi-valued functions like the logarithm (with infinitely many sheets) or the square root (with two sheets) could become one-to-one functions. Complex functions are harmonic functions (that is, they satisfy Laplace's equation and thus the *Cauchy-Riemann equations*) on these surfaces and are described by the location of their singularities and the topology of the surfaces. The topological "genus" of the Riemann surfaces is given by $g = w/2 - n + 1$, where the surface has leaves coming together at branch points. For $g > 1$ the Riemann surface has $(3g - 3)$ parameters (the "moduli").

The famous *Riemann mapping theorem* says that a simply connected domain in the complex plane is "biholomorphically equivalent" (i.e. there is a bijection between them that is holomorphic with a holomorphic inverse) to either the exterior or to the interior of the unit circle. The generalization of the theorem to Riemann surfaces is the famous *Uniformization Theorem*, which was proved in the 19th century by Henri Poincaré and Felix Klein. Here, too, rigorous proofs were first given after the development of richer mathematical tools (in this case, topology). For the proof of the existence of functions on Riemann surfaces, he used a minimality condition, which he called the *Dirichlet principle*.

Weierstrass found a hole in the proof: Riemann had not noticed that his working assumption (that the minimum existed) might not work; the function space might not be complete, and therefore the existence of a minimum was not guaranteed. Through the work of David Hilbert in the Calculus of Variations, the Dirichlet principle was finally established. Otherwise, Weierstrass was very impressed with Riemann, especially with his theory of abelian functions. When Riemann's work appeared, Weierstrass withdrew his paper from Crelle and didn't publish it.

An anecdote from Arnold Sommerfeld shows the difficulties which contemporary mathematicians had with Riemann's new ideas. In 1870, Weierstrass had taken Riemann's dissertation with him on a holiday to Rigi and complained that it was hard to understand. The physicist Hermann von Helmholtz assisted him in the work overnight and returned with the comment that it was "natural" and "very understandable".

Other highlights include his work on abelian functions and $theta\ functions$ on Riemann surfaces. Riemann had been in a competition with Weierstrass since 1857 to solve the Jacobian inverse problems for abelian integrals, a generalization of elliptic integrals. Riemann used theta functions in several variables and reduced the problem to the determination of the zeros of these theta functions. Riemann also investigated period matrices and characterized them through the "Riemannian period relations" (symmetric, real part negative).

Many mathematicians, such as Alfred Clebsch, furthered Riemann's work on algebraic curves. These theories depended on the properties of a function defined on Riemann surfaces.

Real Analysis

In the field of real analysis, he discovered the *Riemann integral* in his habilitation. Among other things, he showed that every piecewise continuous function is integrable. In his habilitation work on Fourier series, where he followed the work of his teacher Dirichlet, he showed that Riemann-integrable functions are "representable" by *Fourier series*. Riemann gave an example of a Fourier series representing a continuous, almost nowhere-differentiable function, a case not covered by Dirichlet. He also proved the *Riemann-Lebesgue lemma*: if a function is representable by a Fourier series, then the Fourier coefficients go to zero for large n.

Number Theory

He made some famous contributions to modern analytic number theory. In a single short paper, the only one he published on the subject of number theory, he investigated the zeta function that now bears his name, establishing its importance for understanding the distribution of prime numbers. The *Riemann hypothesis* was one of a series of conjectures he made about the function's properties.

Publications

Some of his famous writings (which were published after his death) include 'On the Hypothesis Which Lie at the Foundation of Geometry' in 1868, 'Collected Works of Bernhard Riemann' published in 1892, and Collected Papers' published in 2004.

Death

It is said that Riemann caught a cold which worsened to become tuberculosis. Although he made several efforts in order to get better but all in vain. He spent the final days of his life in Italy in the village of Selasca with his wife and daughter. Riemann died on 20th July 1866.

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ZERO

RAJENKI DAS, DEEPIKA SAINI

Abstract. Zero may not be part of the countable numbers family and also, may not have any value on its own but it does increase the value of any countable digit by TEN TIMES. Nevertheless, it has a really interesting history. Zero is not just represented by empty space, it is lot more meaningful than we presume. We may think that zero was the first number to be discovered but you may be surprised to know that it is not true. The history of mathematics is very unclear about the origin of zero. Nobody can, yet, clearly answer who discovered zero. Was it Aryabhatta? The paper explores this and also talks about the different types of zero.

Etymology

The word zero came into the English language via French zèro from Venetian zero, via zefiro from safira or sifr. In pre-Islamic time the word *sifr* (Arabic) had the meaning 'empty'. *Sifr* evolved to mean zero when it was used to translate *ṡũnya* (Sanskrit). The first known use of zero in the English language was in 1598.

The Italian mathematician Fibonacci, who is credited with introducing the decimal system to Europe, used the term *zephyrum.* This became *zefiro* in Italian, and was then contracted to zero in Venetian. The Italian word *zefiro* was already in existence (meaning "west wind" from Latin and Greek *zephyrus*) and may have influenced the spelling when transcribing Arabic *sifr.*

History

Zero was invented independently by the Babylonians, Mayans and Indians (although some researchers say that the Indian number system was influenced by the Babylonians).

- Babylonians:
  Despite the invention of zero as a placeholder, the Babylonians never quite discovered zero as a number. The Babylonians wrote on tablets of unbaked clay, using cuneiform writing. The symbols were pressed into soft clay tablets with the slanted edge of a stylus and so had a wedge-shaped appearance. Many tablets from around 1700 BC survive and we can read the original texts. Of course their notation for numbers was quite different from ours (and not based on 10 but on 60) but to translate into our notation they would not distinguish between 2106 and 216 (the context would have to show which was intended). It was not until around 400 BC that the Babylonians put two wedge symbols into the place where we would put

zero to indicate which was meant, 216 or 21 ” 6.

- Mayans:
  The Mayans, native inhabitants of Central America, were highly skilled mathematicians, astronomers, artists and architects. They had a very complex calendar system and needed a placeholder in their elaborate date system. This led to their invention of zero.

- Indians:
  Zero was used to denote an empty place.A striking note about the Hindu zero is that, unlike the Babylonian and Mayan zero, the Hindu zero symbol came to be understood as meaning “nothing”. This is probably because of the use of number words that preceded the symbolic zero.
  Until 1930, many scholars in the West believed that the zero was either a European or an Arab invention. A highly polemical academic argument was raging at the time, where British scholars, among them G. R. Kaye, who published much about it, mounted strong attacks against the hypothesis that the zero was an Indian invention. The oldest known zero at that time was indeed in India, at the Chatur-bujha temple in the city of Gwalior. But it was dated to the mid-ninth century, an era that coincided with the Arab Caliphate. Thus Kaye’s claim that zero was invented in the West and came to India through Arab traders could not be defeated using the Gwalior zero. But then in 1931, the French archaeologist Georges Cœdès published an article that demolished Kaye’s theory. In it, he proved definitively that the zero was an Eastern (and perhaps Cambodian, although he viewed Cambodia an “Indianized” civilization) invention. Cœdès based his argument on an amazing discovery. Early in the twentieth century, an inscription was discovered on a stone slab in the ruins of a seventh-century temple in a place called Sambor on Mekong, in Cambodia. Cœdès gave this inscription the identifier K-127. He was an expert philologist and translated the inscription from Old Khmer. It begins: *“Chaka parigraha 605 pankami roc...”* which translates to: “The Chaka era has reached 605 on the fifth day of the waning moon...”
  The zero in the number 605 is the earliest zero we have ever found. We know that the Chaka era began in AD 78, so the year of this inscription in our calendar is 605 + 78 = AD 683. Since this time predates the Arab empire, as well as the Gwalior zero, by two centuries, Cœdès was able to prove that the zero is, in fact, an Eastern invention. It is believed to have come to the West via Arab traders and was popularized in Europe through the work of Fibonacci (of the famous sequence of numbers), published in 1202. In 498 AD, Indian mathematician and astronomer Aryabhata stated that *“sthãnãt sthãnaṁ daṡagunaṁ syãt”* i.e., “from place to place each is ten times the preceding” which is the origin of the modern decimal-based place value notation. In around 500 AD, Aryabhata devised a number system which has no zero, yet was a positional system. He used the word *“kha”* for position and

it would be used later as the name for zero. There is evidence that a dot had been used in earlier Indian manuscripts to denote an empty place in positional notation. It is interesting that the same documents sometimes also used a dot to denote an unknown where we might use 'x'. Later Indian mathematicians had names for zero in positional numbers, yet had no symbol for it. The first record of the Indian use of zero which is dated and agreed by all to be genuine was written in 876 AD.

Brahmagupta established the basic mathematical rules for dealing with zero (1 + 0 = 1; 1 - 0 = 1; and 1 x 0 = 0), although his understanding of division by zero was incomplete (he thought that $1 \div 0 = 0$). Almost 500 years later, in the 12th Century, another Indian mathematician, Bhaskara II, showed that the answer should be infinity, not zero (on the grounds that 1 can be divided into an infinite number of pieces of size zero); an answer that was considered correct for centuries. However, this logic does not explain why $2 \div 0$, $7 \div 0$, etc. should also be zero. The modern view is that a number divided by zero is actually "undefined" (i.e. it doesn't make sense). Brahmagupta gave the first rules for dealing with zero as a number: When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero. He also gave arithmetical rules in terms of fortunes (positive numbers) and debts (negative numbers):

- A debt minus zero is a debt.
- A fortune minus zero is a fortune.
- Zero minus zero is a zero.
- A debt subtracted from zero is a fortune.
- A fortune subtracted from zero is a debt.
- The product of zero multiplied by a debt or fortune is zero.
- The product of zero multiplied by zero is zero.
- The product or quotient of two fortunes is one fortune.
- The product or quotient of two debts is one fortune.
- The product or quotient of a debt and a fortune is a debt.
- The product or quotient of a fortune and a debt is a debt.

Brahmagupta then tried to extend arithmetic to include division by zero:-

- Positive or negative numbers when divided by zero is a fraction with the zero as denominator.
- Zero divided by negative or positive numbers is either zero or is expressed as a fraction with zero as numerator and the finite quantity as denominator.
- Zero divided by zero is zero.

Types of zero

It is necessary to distinguish three different types of "zero" : the "intuitive zero", that means "nothing", the "numeral zero" used in the representation of the numbers and the "mathematical zero" defined by the modern mathematicians.

Conclusion

Though zero was found independently in three distinct regions, Indians are considered to be the first ones to invent zero.

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Rigour in Mathematics

This section introduces advance Mathematics to the readers aiming at high standards of proofs. It stimulates interest and lays the foundation for further studies in different branches.

NORMAL CATEGORIES FROM VECTOR SPACES

AZEEF MUHAMMED P A

Abstract. Let T_V be the multiplicative semigroup of all singular linear transformations on an arbitrary vector-space V. It is known that T_V is a regular semigroup. The principal left ideals of a regular semigroup with partial right translations as morphisms form a category $\mathcal{L}(S)$. The category $\mathcal{L}(S)$ is known as the normal category associated with the semigroup S. Every normal category $\mathcal{C}$ gives rise to a regular semigroup $T\mathcal{C}$ of normal cones in $\mathcal{C}$. We show that the semigroup $T\mathcal{L}(T_V)$ of normal cones in $\mathcal{L}(T_V)$ is isomorphic to T_V. The Subspace category $\mathscr{P}(V)$ associated with V is the category whose objects are proper subspaces of V and morphisms are linear transformations. We show that $\mathcal{L}(T_V)$ is isomorphic to the category $\mathscr{P}(V)$.

Keywords : Normal Category, Linear-transformations, Subspace, Cross-connections, Normal Cones.
AMS 2010 Subject Classification : Primary- 20M20, Secondary- 20M17, 20M50, 15A04.

Introduction

A *semigroup* is an abstract algebraic structure consisting of a non-empty set S along with an associative binary operation. A semigroup S is said to be (von-neumann) *regular* if for every $a \in S$, there exists b such that $aba = a$. In the study of the structure theory of regular semigroups, there are mainly two approaches. The first approach inspired by the work of WD Munn (cf. [7]) uses the set of idempotents E of the semigroup to construct the semigroup as a full-subsemigroup of the semigroup of the principal-ideal isomorphisms of E. One of the biggest contributions of India to the world of semigroup theory lies at the heart of this construction wherein KSS Nambooripad (cf. [8]) abstractly characterized the set of idempotents of a (regular) semigroup as a *(regular) biordered set.* It was later proved by D Easdown (cf. [2])that infact the idempotents of any arbitrary semigroup form a biordered set.

The second approach initiated by Hall (cf. [4]) uses the ideal structure of the regular semigroup to analyse its structure. PA Grillet (cf. [3]) refined Hall's theory to abstractly characterize the ideals as *regular partially ordered sets* and constructing the fundamental image of the regular semigroup as a cross-connection semigroup. Again Nambooripad (cf. [9]) generalized the idea to any arbitrary regular semigroups by characterizing the ideals as *normal categories.*

A cross-connection between two normal categories $\mathcal{C}$ and $\mathcal{D}$ is a *local isomorphism* $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ where $N^*\mathcal{C}$ is the normal dual of the category $\mathcal{C}$. A cross-conection Γ determines a cross-connection semigroup $\tilde{S}\Gamma$ and conversely every regular semigroup is isomorphic to a cross-connection semigroup for a suitable cross-connection.

Let T_V be the multiplicative semigroup of all singular linear transformations on an arbitrary vector-space V under composition. It is known that T_V is a regular semigroup. The principal left ideals of a regular semigroup with partial right translations as morphisms form a category $\mathcal{L}(S)$. The category $\mathcal{L}(S)$ is known as the normal category associated with the semigroup S. Every normal category $\mathcal{C}$ gives rise to a regular semigroup $T\mathcal{C}$ of normal cones in $\mathcal{C}$. In this paper, we show that the semigroup $T\mathcal{L}(T_V)$ of normal cones in $\mathcal{L}(T_V)$ is isomorphic to T_V. The Subspace category $\mathscr{P}(V)$ associated with V is the category whose objects are proper subspaces of V and morphisms are linear transformations. We also show that $\mathcal{L}(T_V)$ is isomorphic to the category $\mathscr{P}(V)$.

Preliminaries

We assume familiarity with the definitions and elementary concepts of category theory (cf. [6]). In the following, the definitions and results on normal categories are as in [9]. For a category $\mathcal{C}$, we denote by $v\mathcal{C}$ the set of objects of $\mathcal{C}$.

Definition 0.1. A *preorder* $\mathcal{P}$ is a category such that for any $p, p' \in \mathcal{P}$, the hom-set $\mathcal{P}(p, p')$ contains atmost one morphism.

In this case the relation $\subseteq$ on the class $v\mathcal{P}$ of objects of $\mathcal{P}$ defined by

$$p \subseteq p' \iff \mathcal{P}(p, p') \neq \emptyset$$

is a quasi-order. $\mathcal{P}$ is said to be a strict preorder if $\subseteq$ is a partial order.

Definition 0.2. Let $\mathcal{C}$ be a category and $\mathcal{P}$ be a subcategory of $\mathcal{C}$. Then $(\mathcal{C}, \mathcal{P})$ is called a *category with subobjects* if the following hold:

(1) $\mathcal{P}$ is a strict preorder with $v\mathcal{P} = v\mathcal{C}$.
(2) Every $f \in \mathcal{P}$ is a monomorphism in $\mathcal{C}$.
(3) If $f, g \in \mathcal{P}$ and if $f = hg$ for some $h \in \mathcal{C}$, then $h \in \mathcal{P}$.

In a category with subobjects, if $f : c \to d$ is a morphism in $\mathcal{P}$, then f is said to be an *inclusion*. And we denote this inclusion by $j(c, d)$.

Definition 0.3. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We shall say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *v-injective* if vF is injective. F is said to be *v-surjective* if vF is surjective.

In the following, $(\mathcal{C}, \mathcal{P})$ is a category with subobjects.

Definition 0.4. A morphism $e : d \to c$ is called a *retraction* if $c \subseteq d$ and $j(c, d)e = 1_c$.

Definition 0.5. A *normal factorization* of a morphism $f \in \mathcal{C}(c, d)$ is a factorization of the form $f = euj$ where $e : c \to c'$ is a retraction, $u : c' \to d'$ is an isomorphism and $j = j(d', d)$ for some $c', d' \in v\mathcal{C}$ with $c' \subseteq c$, $d' \subseteq d$.

It may be noted here that normal factorization of a morphism is not unique. But if $f = euj = e'u'j'$ are two normal factorizations of f, then it can be shown that $eu = e'u'$ and $j = j'$. And here we denote eu by f°. Observe that f° is independent of the factorization and that f° is an epimorphism. We call f° the epimorphic part of f.

Definition 0.6. Let $d \in v\mathcal{C}$. A map $\gamma : v\mathcal{C} \to \mathcal{C}$ is called a *cone from the base $v\mathcal{C}$ to the vertex d* (or simply a cone in $\mathcal{C}$ to d) if γ satisfies the following:

(1) $\gamma(c) \in \mathcal{C}(c, d)$ for all $c \in v\mathcal{C}$.
(2) If $c' \subseteq c$ then $j(c', c)\gamma(c) = \gamma(c')$.

Given the cone γ we denote by c_γ the the *vertex* of γ and for each $c \in v\mathcal{C}$, the morphism $\gamma(c) : c \to c_\gamma$ is called the *component* of γ at c.

Definition 0.7. The cone γ is said to be *normal* if there exists $c \in v\mathcal{C}$ such that $\gamma(c) : c \to c_\gamma$ is an isomorphism.

Several normal cones can be derived from a given normal cone. Let σ be a normal cone with vertex d and let $f : d \to d'$ be an epimorphism. Then $\sigma * f$ defined below is a normal cone.

$$(\sigma * f)(a) = \sigma(a)f \tag{1}$$

Definition 0.8. A *normal category* is a pair $(\mathcal{C}, \mathcal{P})$ satisfying the following :

(1) $(\mathcal{C}, \mathcal{P})$ is a category with subobjects.
(2) Any morphism in $\mathcal{C}$ has a normal factorization.
(3) For each $c \in v\mathcal{C}$ there is a normal cone σ with vertex c and $\sigma(c) = 1_c$.

Now we can see that the normal cones in a normal category form a regular semigroup, with product as follows.

Theorem 1. (cf. [9]) Let $(\mathcal{C}, \mathcal{P})$ be a normal category and let $T\mathcal{C}$ be the set of all normal cones in $\mathcal{C}$. Then $T\mathcal{C}$ is a regular semigroup with product defined as follows :
For $\gamma, \sigma \in T\mathcal{C}$.

$$(\gamma * \sigma)(a) = \gamma(a)(\sigma(c_\gamma))^\circ \tag{2}$$

where $(\sigma(c_\gamma))^\circ$ is the epimorphic part of the $\sigma(c_\gamma)$.
Then it can be seen that $\gamma * \sigma$ is a normal cone. $T\mathcal{C}$ is called the *semigroup of normal cones* in $\mathcal{C}$. σ is an idempotent in $T\mathcal{C}$ if and only if $\sigma(c) = 1_c$ where c is the vertex of σ. And if $\gamma, \gamma' \in T\mathcal{C}$, then $\gamma \mathscr{L} \gamma' \iff c_\gamma = c_{\gamma'}$.

Let S be a regular semigroup. The category of principal left ideals of S is described as follows. Since every principal left ideal in S has at least one idempotent generator, we may write objects (vertexes) in $\mathcal{L}(S)$ as Se for $e \in E(S)$. Morphisms $\rho : Se \to Sf$ are right translations $\rho = \rho(e, s, f)$ where $s \in eSf$ and ρ maps $x \mapsto xs$. Thus

$$v\mathcal{L}(S) = \{Se : e \in E(S)\} \quad \text{and} \quad \mathcal{L}(S) = \{\rho(e, s, f) : e, f \in E(S),\ s \in eSf\}. \tag{3}$$

The following proposition gives the general properties of $\mathcal{L}(S)$.

Proposition 1. (cf. [9]) Let S be a regular semigroup. Then

(1) $\mathcal{L}(S)$ is a normal category.
(2) $\rho(e, u, f) = \rho(e', v, f')$ if and only if $e \mathscr{L} e'$, $f \mathscr{L} f'$, $u \in eSf$, $v \in e'Sf'$ and $v = e'u$.
(3) The map $\rho(e, s, f) \mapsto s$ is a bijection of $\mathcal{L}(Se, Sf)$ onto eSf for all $e, f \in E(S)$.

(4) If $\rho(e,u,f)$ and $\rho(e',v,f')$ are composable morphisms in $\mathcal{L}(S)$ (so that $f\mathscr{L}e'$) and $u \in eSf$ and $v \in e'Sf'$, then $\rho(e,u,f)\rho(e',v,f') = \rho(e,uv,f')$.
(5) A morphism $\rho = \rho(e,s,f)$ is a monomorphism [epimorphism] if and only if ρ is injective [surjective]; this is true if and only if $e\mathscr{R}s$ [$s\mathscr{L}f$].
(6) A morphism ρ is the inclusion $Se \subseteq Sf$ if and only if $ef = e$ and $\rho = \rho(e,e,f)$. Then $\rho' : Sf \to Se$ is a retraction if and only if $\rho' = \rho(f,g,e) = \rho(f,g,g)$ where $g \in L_e \cap \omega(f)$.
(7) Let $\rho = \rho(e,u,f)$ be a morphism in $\mathcal{L}(S)$. For any $g \in R_u \cap \omega(e)$ and $h \in E(L_u)$

$$\rho = \rho(e,g,g)\rho(g,u,h)\rho(h,h,f)$$

is a normal factorization of ρ. Every normal factorization of ρ has this form.

Proposition 2. (cf. [9]) Let S be a regular semigroup, $a \in S$ and $f \in E(L_a)$. Then for each $e \in E(S)$, let $\rho^a(Se) = \rho(e, ea, f)$. Then

(1) ρ^a is a normal cone in $\mathcal{L}(S)$ with vertex Sa.
(2) $M_{\rho^a} = \{Se : e \in E(R_a)\}$.
(3) ρ^a is an idempotent in $T\mathcal{L}(S)$ iff $a \in E(S)$.

The normal cone ρ^a is called a principal cone.

Proposition 3. (cf. [9]) If S is a regular semigroup then the mapping $a \mapsto \rho^a$ is a homomorphism from S to $T\mathcal{L}(S)$. Further if S has an identity, then S is isomorphic to $T\mathcal{L}(S)$.

Normal categories and linear transformation semigroup

Let V be an arbitrary vector-space. T_V is the multiplicative semigroup of all singular linear transformations on V. Now we characterize the normal category $\mathcal{L}(T_V)$ associated with the principal left ideals of T_V. The product of transformations is taken in the order it is written .i.e from left to right. In this section we use S and T_V interchangably to denote the semigroup of singular linear transformations on V. Here for any $\alpha \in T_V$, we denote by N_α the null space of α consisting of all $y \in V$ such that $y\alpha = 0$. Since T_V is a regular semigroup, by Proposition 1, $\mathcal{L}(T_V)$ forms a normal category. And we proceed to characterize all the normal cones in this category. The following properties of T_V will be used often.

Lemma 1. (cf. [1]) If $\alpha, \beta \in T_V$, then

(1) $\alpha\mathscr{L}\beta \iff V\alpha = V\beta$.
(2) $\alpha\mathscr{R}\beta \iff N_\alpha = N_\beta$.
(3) $\alpha \in T_V$ is an idempotent $\iff V = N_\alpha \oplus V\alpha$.

Every vector-space has a basis, and let B_A be a basis of A for any proper subspace A of V. $\tilde{A}$ will denote an idempotent transformation (a projection) with range A. We will use SA to denote the principal left ideal of T_V generated by $\tilde{A}$. When $A = \{x\}$, we write Sx for SA and $\tilde{x}$ for $\tilde{A}$.

Lemma 2. Let $A, B \subseteq V$ and $\rho(\tilde{A}, \alpha, \tilde{B})$ be a morphism from SA to SB. Then for any $x \in A$, $x\alpha \in B$. Also if $\rho(\tilde{A}, \alpha, \tilde{B})$, $\rho(\tilde{A}, \beta, \tilde{B})$ are morphisms from SA to SB, then $\rho(\tilde{A}, \alpha, \tilde{B}) = \rho(\tilde{A}, \beta, \tilde{B})$ if and only if $b\alpha = b\beta \quad \forall b \in B_A$.

Proof. By the definition of a morphism in $\mathcal{L}(T_V)$, $\alpha \in \tilde{A}S\tilde{B}$. So by lemma 1, $\mathrm{V}\alpha \subseteq \mathrm{V}\tilde{B} = B$. Hence $x\alpha \in B$ for any $x \in V$ and in particular for any $x \in A$.
To prove the second part, if $\rho(\tilde{A}, \alpha, \tilde{B}) = \rho(\tilde{A}, \beta, \tilde{B})$, then by Proposition 1, $\tilde{A}\alpha = \tilde{A}\beta$. And so $x\alpha = x\beta \quad \forall x \in A$ and in particular for $b \in B_A$. Conversely if $b\alpha = b\beta \quad \forall b \in B_A$, since a linear transformation on A is completeley detrmined by the values on its basis, we have $\tilde{A}\alpha = \tilde{A}\beta$ and so, we have $\rho(\tilde{A}, \alpha, \tilde{B}) = \rho(\tilde{A}, \beta, \tilde{B})$. □

Proposition 4. All normal cones in the category $\mathcal{L}(T_V)$ are the principal cones.

Proof. Given $\alpha \in S$, then we know that ρ^α is a normal cone in $\mathcal{L}(S)$(by Proposition 2). Now we need to show that every normal cone is principal. Let σ be a normal cone in $\mathcal{L}(T_V)$, then $c_\sigma = SA$ for some $A \subseteq V$. Consider $\sigma(Sb)$ for $b \in B_V$. Let $\sigma(Sb) = \rho(\tilde{b}, u_b, \tilde{A})$. By Lemma 2, $bu_b \in A$. Define α on V as follows:

$$b\alpha = bu_b \quad \forall b \in B_V \text{ where } u_b \text{ is as above.}$$

Clearly α is well-defined. Since $bu_b \in A \quad \forall b \in B_V$, and a linear transformation on V is completely determined by the values on the basis elements, α is a linear transformation from V with image contained in A. Since σ is a normal cone with vertex SA, there is a component $\sigma(SC)$ such that $\sigma(SC)$ is an isomorphism. Let $\sigma(SC) = \rho(\tilde{C}, \beta, \tilde{A})$. Then by Lemma 2, $b\beta \in A$ for all $b \in B_C$. Since $\sigma(SC)$ is an isomorphism, Im $\beta = A$ by Proposition 1.
Now we show that Im $\alpha = A$. Let $y \in A$. Then there exists $b \in B_C$ such that $b\beta = y$. Now $\rho(\tilde{b}, u_b, \tilde{A}) = \sigma(Sb) = j(Sb, SC)\sigma(SC) = j(Sb, SC)\rho(\tilde{C}, \beta, \tilde{A})$.
Therefore $\rho(\tilde{b}, u_b, \tilde{A}) = \rho(\tilde{b}, \tilde{b}, \tilde{C})\rho(\tilde{C}, \beta, \tilde{A}) = \rho(\tilde{b}, \tilde{b}\beta, \tilde{A})$ (by Proposition 1).
And $u_b = \tilde{b}\beta$ so that $b\alpha = bu_b = b(\tilde{b}\beta) = b\beta = y$.
Hence α is onto A. Now we show that $\sigma = \rho^\alpha$
Since Im $\alpha = A$, we see that the vertex of ρ^α is $SA = c_\sigma$.
So the cones σ and ρ^α have the same vertex. Now we show that for $D \subseteq V$, if $\sigma(SD) = \rho(\tilde{D}, \delta, \tilde{A})$; then $\rho(\tilde{D}, \delta, \tilde{A}) = \rho(\tilde{D}, \tilde{D}\alpha, \tilde{A})$.
For that by Lemma 2, it is sufficient to prove that $b\delta = b\alpha \; \forall b \in B_D$.
If $b \in B_D$, then $Sb \subseteq SD$ and by the definition of the normal cones, $\sigma(Sb) = j(Sb, SD)\sigma(SD) = \rho(\tilde{b}, \tilde{b}, \tilde{D})\rho(\tilde{D}, \delta, \tilde{A}) = \rho(\tilde{b}, \tilde{b}\delta, \tilde{A})$. But $\sigma(Sb) = \rho(\tilde{b}, u_b, \tilde{A})$. Equating these we get for $b\tilde{b}\delta = bu_b$. So $b\delta = bu_b$ (since $b\tilde{b} = b$) .i.e $b\delta = b\alpha \quad \forall b \in B_D$.
Therefore $\rho(\tilde{D}, \delta, \tilde{A}) = \rho(\tilde{D}, \tilde{D}\alpha, \tilde{A})$.
Hence $\sigma = \rho^\alpha$. Thus all normal cones are of the form ρ^α for some $\alpha \in S$. □

In general, for a regular semigroup S, and the associated normal category $\mathcal{L}(S)$, the semigroup of normal cones in $\mathcal{L}(S)$ is not isomorphic to S. By Proposition 3, we have a homomorphism from S to $T\mathcal{L}(S)$ which may not be one-one or onto. But in the case of T_V, we indeed have an isomorphism.

Theorem 2. $T\mathcal{L}(T_V)$ is isomorphic to T_V.

Proof. Consider the map $\alpha \mapsto \rho^\alpha$ from T_V to $T\mathcal{L}(T_V)$. This map is a semigroup homomorphism (By Proposition 3). And by Proposition 4, the map is onto. Now we need to show that it is 1-1. Let $\alpha, \beta \in T_V$ such that $\rho^\alpha = \rho^\beta$. Then for any $b \in B_V$, $\rho^\alpha(Sb) = \rho(\tilde{b}, \tilde{b}\alpha, \epsilon)$ where $\epsilon \in E(L_\alpha)$ and $\rho^\beta(Sb) = \rho(\tilde{b}, \tilde{b}\beta, \delta)$ where $\delta \in E(L_\beta)$.
And since $\rho^\alpha = \rho^\beta$, we have $\rho(\tilde{b}, \tilde{b}\alpha, \epsilon) = \rho(\tilde{b}, \tilde{b}\beta, \delta)$ and so $\tilde{b}\alpha = \tilde{b}\beta$. It follows that $b\alpha = b\beta$ for all $b \in B_V$. And $\alpha = \beta$.
Hence the result. □

0.1. **The category of subspaces of V.** Now we show that the category $\mathcal{L}(T_V)$ can be identified with the category of proper subspaces of V- the Subspace category. It is easy to see that all the proper subspaces of a vector-space V with linear transformations as morphisms forms a category $\mathscr{P}(V)$ (cf. [11]). Observe that $\mathscr{P}(V)$ has an obvious choice of subobjects since the subspace-inclusion of $A \subseteq B$ gives a natural preorder in $\mathscr{P}(V)$. And composition of two morphisms is composition. Now we proceed to show that this category is a normal category.

Proposition 5. $\mathscr{P}(V)$ is a normal category.

Proof. The inclusions in $\mathscr{P}(V)$ are precisely inclusion transformations .i.e we have an inclusion $j : A \to B$ if and only if $A \subseteq B$ as subspaces of V. $q : B \to A$ is a retraction if and only if $A \subseteq B$ and $j(A, B)q = 1_A$. It is easy to see that any linear transformation f from A to B has a factorization $f = euj$ where $e : A \to A'$ is a retraction, $u : A' \to B'$ is a vector-space isomorphism and $j = j(B', B)$ is an inclusion when $B' = \operatorname{Im} f$ and A' is a cross-section of the partition of A determined by f. Given any $A \subseteq V$, let σ be a cone in $\mathscr{P}(V)$ with vertex A defined as follows. Let $u : V \to A$ be a linear transformation such that $u(b) = b \quad \forall b \in B_A$. For any $B \subseteq V$, define $\sigma(B) = u_{|B} : B \to A$. Then σ is a normal cone with $\sigma(A) = 1_A$. Hence $\mathscr{P}(V)$ is a normal category. □

Now we proceed to show that $\mathscr{P}(V)$ is isomorphic to the category of principal left ideals of T_V as normal categories. For that, we show that there exists an inclusion preserving functor from $\mathcal{L}(T_V)$ to $\mathscr{P}(V)$ which is an order isomorphism, v-injective, v-surjective and fully-faithful.
We define a functor $F : \mathcal{L}(T_V) \to \mathscr{P}(V)$ as follows. For $SA = T_V A \in v\mathcal{L}(T_V)$ and a morphism $\rho(\tilde{A}, \alpha, \tilde{B})$ in $\mathcal{L}(T_V)$

$$vF(SA) = A \quad \text{and} \quad F(\rho(\tilde{A}, \alpha, \tilde{B})) = \alpha_{|A}. \tag{4}$$

By Lemma 2, α maps vectors of A to vectors of B. So the restriction of α to A gives a well-defined linear transformation from A to B. Now we show F is an inclusion preserving isomorphism of normal categories. For that first we need to show that F is functorial.

Lemma 3. F as defined in equation 4 is a well defined functor from $\mathcal{L}(T_V)$ to $\mathscr{P}(V)$.

Proof. vF is well-defined by Lemma 1. Now if $\rho(\tilde{A}, \alpha, \tilde{C}) = \rho(\tilde{B}, \beta, \tilde{D})$. Then by Proposition 1 and Lemma 1 , we get $\tilde{A} \mathscr{L} \tilde{B}$; and hence $A = B$. And similarly $C = D$.

Now $\alpha_{|A} = \tilde{A}\beta_{|A}$ (By Proposition 1)
$= \beta_{|B}$ since $A = B$ and $\tilde{A}$is an idempotent; we have $\tilde{A}_{|A} = 1_A$.
Hence F is well defined on morphisms as well.
Now let $\rho(\tilde{A}, \alpha, \tilde{B})$, $\rho(\tilde{B}, \beta, \tilde{C})$ be two composable morphisms in $\mathcal{L}(T_V)$; then by Proposition 1, $F(\rho(\tilde{A}, \alpha, \tilde{B})\rho(\tilde{B}, \beta, \tilde{C})) = F(\rho(\tilde{A}, \alpha\beta, \tilde{C})) = \alpha\beta_{|A}$.
Also $F(\rho(\tilde{A}, \alpha, \tilde{B}))F(\rho(\tilde{B}, \beta, \tilde{C})) = \alpha_{|A}\beta_{|B} = \alpha\beta_{|A}$ since $\alpha_{|A} : A \to B$ and $\beta_{|B} : B \to C$.
Hence F is functorial. □

Lemma 4. F is inclusion preserving.

Proof. Suppose $SA \subseteq SB$; then by Lemma 1 and Proposition 1, $A \subseteq B$.
Also $F(j(SA, SB)) = F(\rho(\tilde{A}, \tilde{A}, \tilde{B})) = \tilde{A}_{|A} = j(A, B)$.
Thus F is inclusion preserving. □

Lemma 5. vF is an order isomorphism.

Proof. Suppose $SA \subseteq SB$. Then by lemma 1 and Proposition 1, $A \subseteq B$.
Conversely if $SA \subseteq SB$, then $A \subseteq B$.
Hence $SA \subseteq SB \iff A \subseteq B$ and so vF is an order isomorphism. □

Lemma 6. F is v-surjective and full.

Proof. Let $A \subsetneq V$ then clearly there exists an idempotent transformation in T_V say $\tilde{A}$ such that the Im $\tilde{A} = A$. Now $F(SA) = A$. Hence F is v-surjective.
Now let f be function from A to B and let $\alpha = \tilde{A}f$. Then α is a full transformation with image B and $\alpha_{|A} = f$. So $\alpha \in \tilde{A}S\tilde{B}$.
And $F(\rho(\tilde{A}, \alpha, \tilde{B})) = \alpha_{|A} = f$.
Hence F is full. □

Lemma 7. F is v-injective and faithful.

Proof. Let $\tilde{A}$ and $\tilde{B}$ denote idempotent transformations with image A and B respectively.
And let $F(S\tilde{A}) = F(S\tilde{B})$ in $\mathscr{P}(V)$. Then $A = B$. And by lemma 1, $S\tilde{A} = S\tilde{B}$. Hence F is v-injective.
Now let $\rho(\tilde{A}, \alpha, \tilde{C})$ and $\rho(\tilde{A}, \beta, \tilde{C})$ be morphisms from $S\tilde{A}$ to $S\tilde{C}$ in $\mathcal{L}(T_V)$.
And let $F(\rho(\tilde{A}, \alpha, \tilde{C})) = F(\rho(\tilde{A}, \beta, \tilde{C}))$ in $\mathscr{P}(V)$.
Then $\alpha_{|A} = \beta_{|A}$ so that by Lemma 2, we get $\rho(\tilde{A}, \alpha, \tilde{C}) = \rho(\tilde{A}, \beta, \tilde{C})$.
And hence F is faithful. □

Theorem 3. $\mathcal{L}(T_V)$ is isomorphic to $\mathscr{P}(V)$ as normal categories.

Proof. By the previous lemmas 3, 4, 5, 6, 7 ; F is an inclusion preserving functor from $\mathcal{L}(T_V)$ to $\mathscr{P}(V)$ which is an order isomorphism, v-injective, v-surjective and fully-faithful.
Hence the theorem. □

Since $\mathscr{P}(V)$ forms a normal category, the semigroup of normal cones of $\mathscr{P}(V)$ will form a regular semigroup. Now we prove that it is T_V.

Theorem 4. T_V is isomorphic to the semigroup of normal cones in $\mathscr{P}(V)$.

Proof. By Theorem 3, $\mathscr{P}(V)$ is isomorphic to $\mathcal{L}(T_V)$ as normal categories. So the semigroup of normal cones associated with these two categories will be isomorphic to each other. Hence $T\mathscr{P}(V)$ is isomorphic to $T\mathcal{L}(T_V)$ as semigroups. But by Theorem 2, $T\mathcal{L}(T_V)$ is isomorphic to T_V. Hence $T\mathscr{P}(V)$ is isomorphic to T_V as semigroups. □

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MULTI-OBJECTIVE APPROACH FOR OPTIMISATION OF AN ECO-EFFICIENT REVERSE LOGISTICS NETWORK

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Abstract. One of the key issues concerning manufacturers is designing a reverse logistics (RL) network which goes beyond just cost reduction and customer satisfaction. The aim of this study is to adopt a sustainable approach for designing a reverse logistics network for End-of-life (EOL) and End-of-use (EOU) electronic products which can also generate profit for the company. The purpose of the research is to investigate how electronic product returns are handled in a RL network and which recovery options are most feasible and effective. It also determines the optimal routes for flow of products and selection of vehicles for transportation.

Introduction

Due to rapid development in technology, electronic products have a short life cycle and soon become obsolete but still retain some value which can be recovered. Some products are recalled but still can be refurbished to be resold again. There are end of life products which can be dismantled for component recovery and recycled for material recovery. The manufacturer can choose from the various recovery options as per the quality of the returns to recovery maximum value from EOL and EOU electronic returns in the most sustainable way. However, an efficient coordination is needed amongst all the facilities of the RL network which can be obtained with an effective design of the network. The management of the recovery process also requires an efficient transportation network for collection of the returns and redistribution of the refurbished products and components. The major contribution towards carbon emissions comes from the transportation activities of the network and the emissions level is based on the distance travelled between facilities. In addition, the transportation cost contributes to a major part of the total cost of the network. Therefore to maximise the revenue generated, the transportation distance must also be minimised and to mitigate the impact of transportation on the environment, unnecessary transport activities must be avoided and appropriate vehicle selection must be done.

Model description

The proposed network [2] consists of a collection zone consisting of collection/ inspection centres (CICs), fabrication centre (FC), integrated dismantling centre (DMC) and component fabrication centre (CFC), secondary markets, service centre, spare market, recycling centre and disposal centre as shown in Figure 1. The flow of returned products across various echelons depends on value added recovery options as well as their demand in the secondary markets. Depending on the condition of the returned products, they are classified into two categories at CICs: refurbishable and non refurbishable and are accordingly transported to either FC or DMC. The products which are refurbished at FC are sold at the secondary markets where the demands of these products are high owing to large price differential between the old and the new products. At DMC, the products are dismantled and their components are further classified into three categories based on whether they can be fabricated, recycled or disposed. The fabricated components are used at FC for refurbishing of the product, at the service centre for repairing the products and after satisfying the demands of both the FC and the service center, the remaining fabricated components are sold in the spare markets. The components move internally from the DMC to CFC where the components are fabricated for further use. The fabricated components are transported from CFC to service centre, fabrication centre or spare markets along fixed routes and are carried by appropriately selecting a vehicle based on its capacity and the amount to be transported. Transportation of components to the recycling centre and the disposal centre is managed externally by agents. The recycling agent pays a certain revenue for the components and material collected for recycling but the disposal cost is borne by the manufacturer who pays the agent a fixed per unit revenue for proper disposal.

Assumption

Locations and capacities of CIC_s are known and fixed; locations of DMC, FC and disposal center are known and fixed; demand of the fabricated products and components are known; there is no holding of inventory at any facility; the estimated emission rates of CO_2 for available vehicles are known.

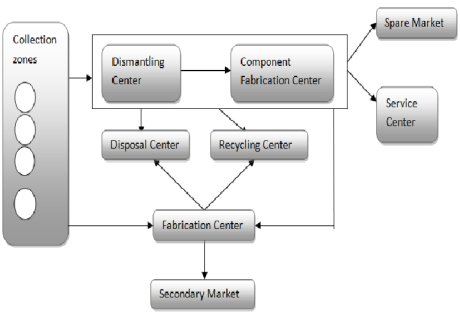

Figure 1: RL Network Design

Mathematical Model

Sets

Set of products indexed by p, set of components indexed by a, set of CICs indexed by l, set of secondary markets indexed by m, set of small trucks indexed by t. Ir and Irare sets of nodes in the r^{th} and r^{th} cluster constructed for flow of products for fabrication and dismantling respectively, $Vr = Ir \cup \{0\}$ where 0 represents FC and $Vr' = Ir' \cup \{0'\}$ where 0' represents DMC.

Parameters

CICn , CDMn and CFCn are per unit cost of inspection, dismantling and fabrication of n^{th} product respectively. CCFa and CDCa are per unit cost of fabrication and disposal of d^{th} component respectively. TC is the transportation cost (per km) of the returned products transported from CIC to DMC and TS is the unit transportation cost of the disassembled components transported from DMC to disposal center, TI is the unit transportation cost of the fabricated components transported from CFC to FC, TK is the unit transportation cost of the fabricated components transported from CFC to service center, CH_t and CL_t are the costs associated with carbon emission of the t^{th} truck for transporting components from CFC along Route 1 and Route 2 resp. RX_{nl} are the units of n^{th} product collected at l^{th} CIC, SDEMa and FDEMa are the demands of the d^{th} component at the service center and at the FC, Q_{na} takes value 1 if n^{th} product consists of d^{th} component, otherwise zero, ρ_n is the fraction of total units of the n^{th} product collected at CICs to be transported for fabrication,$\mu_a, \gamma_a, \delta_a, \omega_a$ are fractions of total units of the d^{th} component transported from FC to recycling centre, transported from DMC to CFC, transported from DMC to recycling center and transported from CFC to service centre respectively. ,DEMmn and PREVmn are the demands of and the revenue generated by the n^{th} product at m^{th} secondary market respectively. FREVa , CREVa , RREVa and SREVa are revenues generated by d^{th} component at FC, spare market, recycling center and service center respectively. is the distance between nodes ir and jr of the r^{th} cluster set V_r, and is the distance between nodes ir and jr of the r^{th} cluster set V_r. , are real numbers used to avoid subtouring in the routes within each cluster. wta is the weight of a^{th} component. CO_2k and CO_2t are the carbon emission per km of the trucks used for product flow and component flow resectively.Route1 and Route2 are the distances travelled from CFC to FC to service centre and from CFC to spare market respectively.

Variables

Z_n and O_n are the units of n^{th} product transported from CICs to DMC and from CICs to FC resp , Bmn are the number of returned units of nth product transported from FC to m^{th} secondary market, G_a, V_a , DISa, RKa, RIa and RFa are units of a^{th} component transported to CFC, recycling center, disposal center, from CFC to service center, CFC to FC and from CFC to spare market respectively. α_a takes value 1 if the number of units of a^{th} component transported to the service center do not exceed the demand , else takes value 0, β_a takes value 1 if the number of units of a^{th} component transported to the FC do not exceed the demand else takes value 0. takes value 1 if the truck travels from node ir to node jr of cluster sets Vr, otherwise zero, takes value 1 if the truck travels from node ir to node jr of cluster sets V_r, otherwise zero . H_t takes value 1 if vehicle t is selected for Route1 otherwise zero, Lt takes value 1 if vehicle t is selected for Route 2 otherwise zero, R is the total weight of the components transported via Route1 and RS is the total weight of the components transported via Route2.

Clustering Algorithm

We propose a two stage algorithm:

- To cluster nodes (CICs), based on proximity and determine the number of CICs each truck will visit in each cluster.
- To determine the optimal route for each truck using TSP

K- means Clustering k-means algorithm is the most commonly used clustering techniques. The k-means algorithm takes the input parameter, k, which is the number of clusters to be formed. Staring with randomly selecting k of the data points as the cluster means, L data points are assigned to the clusters based on its proximity with the cluster means. New means for each cluster are computed and the process iterates until the value of the criterion function converges.

Description: There are total L numbers of CICs which are the nodes. Clustering is based on the distance measure and the total amount of returns collected for fabrication /dismantled per cluster not exceeding the truck capacity. The steps are as follows:

- The number of clusters N1 and N2 for transportation of products to be fabricated and dismantled respectively are calculated as:
  N1(or N2)= [total returns to be fabricated(or dismantled)/ truck capacity where [x] is the smallest integer greater than equal to x
- Choose N1 and N2 arbitrary cluster means m_j , m'_j
- Assign data points to clusters as follows:
  $w_l \in C_k$ iff $\|w_l - m_k\|^2 \leq \|w_l - m_p\|^2 \quad \forall p \in N1.and \sum_{w_l \in C_k} \sum_n \rho_n RX_{nl} \leq truckcapacity$
  $w_l \in C'_k$ iff $\|w_l - m'_k\|^2 \leq \|w_l - m'_p\|^2 \quad \forall p \in N2.and \sum_{w_l \in C'_k} \sum_n (1-\rho_n) RX_{nl} \leq truckcapacity$
- Calculate the value of the criterion functions,
  $E_{FC} = \sum_{l=1}^{L} \sum_{p=1}^{N1} Y_{lp} \|w_l - m_p\|^2$ and $E_{DC} = \sum_{l=1}^{L} \sum_{p=1}^{N2} S_{lp} \|w_l - m'_p\|^2$
  where $Y_i p$ and $S_i p$ are 1 if the $l^t h$ data pointr assigned $p^t h$ cluster, else 0.
- For each cluster, the new mean is computed as follows:
  $m_p* = 1/|C_p| \sum_{w_l \in C_p} w_l$ and $m'_p* = 1/|C'_p| \sum_{w_l \in C'_p} w_l$ where —c_p— in the number of nodes in cluster p
- For the new set of cluster means, assign the data points to the clusters.
- Calculate the value of the criterion functions for the new clusters.
- Repeat steps 5 to 7 until the criterion functions converge.

Problem Formulation

Problem P1:
Objective : Maximize profit

$$\sum_m \sum_n PREV_{mn} B_{mn} + \sum_a \{SREV_a RK_a + RREV_a + FREV_a RI_a + CREV_a RF_a\} -$$

$$\sum_a \{TSDIS_a + TKRK_a + TIRI_a\} - \sum_t \{CH_t H_t Routel + CL_t L_t Routel2\} -$$

$$TC[\sum_r \sum_i \sum_j \{d_{ir}jrx_{ir}jr + d_{ir}jrx_{ir}jr\}] - \sum_n [CIC_n \sum_l RX_{nl} + CDM_n Z_n CFC_n B_{mn}] -$$

$$\sum_a (CCF_a G_a + CDC_a DIS_a)$$

The objective intents to maximize profit which is the difference between the total revenue generated and the various costs namely transportation cost and cost incurred at various facilities while selecting trucks for transportation of products and components across facilities with least carbon emission factor per km. subject to:

$$O_n = \sum_l \rho_n RX_{nl} \quad \forall n \tag{1}$$

$$\sum_m B_{mn} \leqslant O_n \quad \forall n \tag{2}$$

$$B_{mn} \leqslant DEM_{mn} \quad \forall m, n \tag{3}$$

$$Z_n = \sum_l (1 - \rho_n) RX_{nl} \quad \forall n \tag{4}$$

$$G_a = \sum_n \gamma_a Q_{na} Z_n \quad \forall a \tag{5}$$

$$V_a = \sum_n \delta_a Q_{na} Z_n \quad \forall a \tag{6}$$

$$DIS_a = \sum_n (1 - \gamma_a - \delta_a) Q_{na} Z_n \quad \forall a \tag{7}$$

$$RK_a \leqslant \omega_a G_a \quad \forall a \tag{8}$$

$$RI_a \leqslant \phi_a G_a \quad \forall a \tag{9}$$

$$RK_a \geqslant SDEM_a \quad \forall a \tag{10}$$

$$RI_a \geqslant FDEM_a \quad \forall a \tag{11}$$

$$RF_a = (G_a - RK_a - RI_a) \quad \forall a \tag{12}$$

$$\sum_{jr=0}^{Ir} x_{jr}jr = 1 \quad \forall ir \neq jr, ir \in Vr; r = 1, 2...N_1 \tag{13}$$

$$\sum_{ir=0}^{Ir} x_{ir}jr = 1 \quad \forall ir \neq j_r, jr \in Vr; r = 1, 2...N_1 \tag{14}$$

$$e_{jr} - e_{ir} \geq (I_r + 1)x_{ir}jr - I_r \quad \forall ir \neq j_r, ir \in I_r; \tag{15}$$

$$\sum_{jr=0}^{Ir'} x_{ir}jr' = 1 \quad \forall ir \neq j_r, ir \in Vr'; r = 1, 2...N_2 \tag{16}$$

$$\sum_{ir=0}^{Ir'} x_{ir}jr' = 1 \quad \forall ir \neq j_r, jr \in Vr'; r = 1, 2...N_2 \tag{17}$$

$$e_{jr} - e_{ir'} \geq (I'_r + 1)x_{ir}jr' - I'_r \quad \forall ir \neq j_r, ir \in I'_r; \tag{18}$$

$$R = \sum_a (RK_a + RI_a)wt_a \tag{19}$$

$$RS = \sum_a RF_a wt_a \tag{20}$$

$$R \leq \sum_t CAP_t H_t \tag{21}$$

$$\sum_t H_t = 1 \tag{22}$$

$$RS \leq \sum_t CAP_t L_t \tag{23}$$

$$\sum_t L_t = 1 \tag{24}$$

$$B_{mn}, Z_n, O_n, V_a, G_a, DIS_a, RI_a, Rk_a, RF_a \geq 0 \quad \forall m, n, a \in Integer \tag{25}$$

$$H_t, L_t, x_{ij}jr, k'_{ir}jr \in \{0,1\} \quad \forall ir, a, jr, t \tag{26}$$

The constraints (1)-(3) determine the total number of units of each product transported to fabrication center after initial inspection and after realizing the total demand at the secondary markets. The constraints (4)-(12) determine the volume flow of products/ components at various facilities while satisfying the demand of service and fabrication center. The constraints (13)-(18) determines the transportation between the clusters of collection centers using travelling salesman problem to minimize the distance travelled for the transportation of products to FC and DMC respectively. The constraints (19)-(24) determine the selection of vehicle for component flow. Constraint (25) and (26) determine non negativity and binary variables.

Case study

The above model can be efficiently utilised by a company wanting to adopt a more sustainable approach to improve the economical and environmental performance of their RL network. Here we consider a case of an electronics manufacturing firm which has 12 collection centers, a dismantling center and a fabrication center and a service center. There are 2 secondary markets, and a spare market where the refurbished products and components can be sold respectively. A fleet of homogenous vehicles of capacity 2500kg for distribution of returned products and four vehicles of varying capacities for distribution of fabricated components are also available. The company manufactures 4 variants of both air conditioners and refrigerators of varying weights. Their major components are: Compressor (A1), Condenser (A2), Orifice Tube/ Expansion Valve (A3), Evaporator (A4), Accumulator/ Drier (A5), Refrigerant (A6), Valve (A7), Compressor Clutch (A8), Refrigerant Oil (A9), Hose Assembly (A10), Switch (A11), Control Panel (A12), Metal and plastics (A13). All components are common to both except A5, A7, A8, A9, A10 and A12 which are only in AC. The manufacturer would bear transportation cost (per km) of ‘50 for the returned products to and fro FC/DC to CICs, Unit transportation cost(per component) of '.7 from DC to disposal centers, of ' 8 from CFC to FC and of ' 7 from CFC to service center. The cost is based on various factors such as weight of the unit, loading and deloading. Cost of the tth truck for transporting components from the CFC is ' 20,30,40,25 respectively. The value of CIC is '70, '60, '80, '90 and '80, '70, '70, '60 CDM is '70, '95, '80, '90 and '80, '100 , '80, '90, CFC is '110, '115, '120, '135 and '140, '115, '120, '125 for ACs and refrigerators and their weights are 25kg, 30kg, 40kg, 50kg and 50kg, 70kg, 85 kg, 110kg respectively. We assume the fraction of returned products to be fabricated as 0.52, 0.5, 0.53, 0.55 and 0.56, 0.5, 0.54, 0.5 for ACs and refrigerators respectively. The revenue, demand of ACs from secondary market1 is '4000,4500,5000,5400 and 8,9,4,3 while for refrigerators the revenue and demand are '5000, 5400,5800, 7000 and 4,3,2,1 from secondary market2 is '4000,4500,5000,5400 and 10,8,5,3 while for refrigerators the revenue and demand are '5000, 5400,5800, 6200 and 4,5,3,2. The maximum distance allowed for route1 is 60km and route2 is 40km. The capacities of the four trucks 940, 1880, 2000, 2500 kg.

Result

The proposed model is validated using the above data and solved using LINGO11.0. The location of FC, DMC and CIC are shown in figure 2 while figure 3 shows the clusters of CICs based on the distance between the CICs and FC or DMC and the capacity of the vehicle clusters as shown in figure 2 are obtained. Figure 4 and 5 given below show the optimal routes within each clusters for the flow of products.

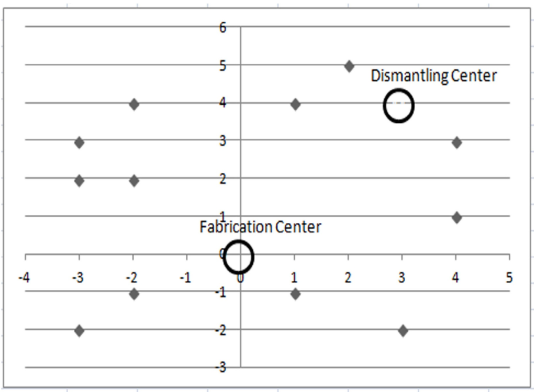

Figure 2: Location of CIC, FC and DMC

For flow of components routes were fixed. For route 1, truck 1 is selected (weight of components 676kg) and for route 2, truck 3 is selected (weight of components 2710kg). For the given set of data, the optimal profit obtained in the network '553452.4. About 60

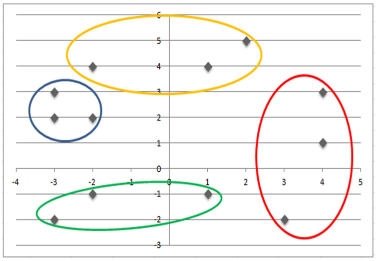

Figure 3: Clusters of CICs

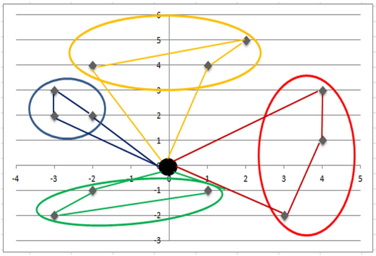

Figure 4: Optimal Routes with respect to FC

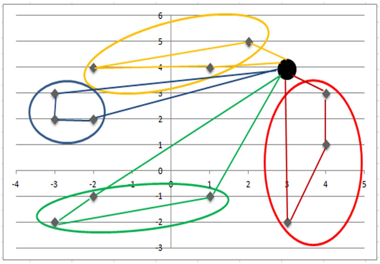

Figure 5: Optimal Routes with respect to DMC

Conclusion

A network designed for product recovery is economically viable if the manufacturer takes full responsibility of the entire life cycle of the product and the returned product moves only in the direction of value adding recovery. The model demonstrates the impact of demand of fabricated products and components on the economic potential of the RL network. Fabricated components are also utilised for fabrication of the products resulting in cost savings. The paper focuses on the appropriate choice of recovery options as per the state of the returned products to derive maximum utility which implicitly leads to profit and on the planning of suitable routes for flow of returns to minimise the distance between the facilities and therefore the carbon emission.

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ROBE'S RESTRICTED PROBLEM OF 2+2 BODIES WITH OBLATE PRIMARIES

BHAVNEET KAUR

Abstract. In this problem, one of the primaries of mass m_1^* is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1. The smaller primary of mass m_2 is an oblate body outside the shell. The third and the fourth bodies (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the shell, with the assumption that the mass and the radius of the third and the fourth body are infinitesimal. We assume that m_2 is describing a circle around m_1^*. The masses m_3 and m_4 mutually attract each other, do not influence the motions of m_1^* and m_2 but are influenced by them. We also assume that masses m_3 and m_4 are moving in the plane of motion of mass m_2. In the paper, equilibrium solutions of m_3 and m_4 and their linear stability are analyzed.

Introduction

[?] has investigated a new kind of restricted three-body problem in which one of the primaries is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1. The mass of the shell including the mass of the fluid is m_1^*. The second primary is a mass point m_2 outside the shell. The third body of mass m_3, supposed moving inside the shell, is a small solid sphere of density ρ_3, with the assumption that the mass and the radius of the third body are infinitesimal. He further assumed that the mass m_2 describes a Keplerian orbit around the mass m_1^*. He has proved that the point $(-\mu, 0)$, the centre of the first primary, is the only equilibrium solution for all values of the density parameter K, mass parameter μ, eccentricity parameter e.[?] have proved that besides $(-\mu, 0)$, there are other equilibrium solutions.

Celestial bodies in general are not spherical, rather they are oblate or axis symmetric bodies. It is therefore essential that we concentrate on primaries which are axis symmetric bodies and preferably on oblate bodies. Many authors have worked taking primaries as oblate bodies. [?] have studied the effect of oblateness on the location and stability of equilibrium solutions in Robe's circular problem.

Many authors have worked on problem of 2+2 bodies. Significant work is that of [?]. He studied equilibrium solutions of the restricted problem of 2+2 bodies. He further studied the linear stability of all the equilibrium solutions.

In this paper, we shall study the case of the restricted problem of 2+2 bodies in the Robe's setup when the smaller primary is an oblate body.

Statement of the Problem and Equations of Motion

In this problem , one of the primaries of mass m_1^* is a rigid spherical shell filled with homogeneous incompressible fluid of density ρ_1. The second primary of mass $m_2(m_1^* > m_2)$ is an oblate body outside the shell. The third and the fourth body (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the shell, with the assumption that the mass and radius of the third and the fourth body are infinitesimal. We assume that m_2 is moving around m_1^* with angular velocity $\boldsymbol{\omega}$ (say) in a circular orbit of radius a'. The masses m_3 and m_4 mutually attract each other but do not influence the motions of m_1^* and m_2. We also assume that masses m_3 and m_4 are moving in the plane of motion of mass m_2.

Let the orbital plane of m_2 around m_1^* is taken as the $\xi\eta$ plane and the origin of the co-ordinate system is at the centre of mass O of the two finite bodies. The coordinate system $O\xi\eta$ is as shown in the Figure 1. Let the synodic system of coordinates initially coincident with the inertial system rotate with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1^*. Let initially the principal axes of m_2 be parallel to the synodic axes and their axes of symmetry be perpendicular to the plane of motion. Since m_2 is revolving without rotation about m_1^* with the same angular velocity as that of the synodic axes, the principal axes of m_2 will remain parallel to them throughout the motion.

Let the coordinates of m_3 and m_4 be (ξ, η) and (ξ', η') respectively.

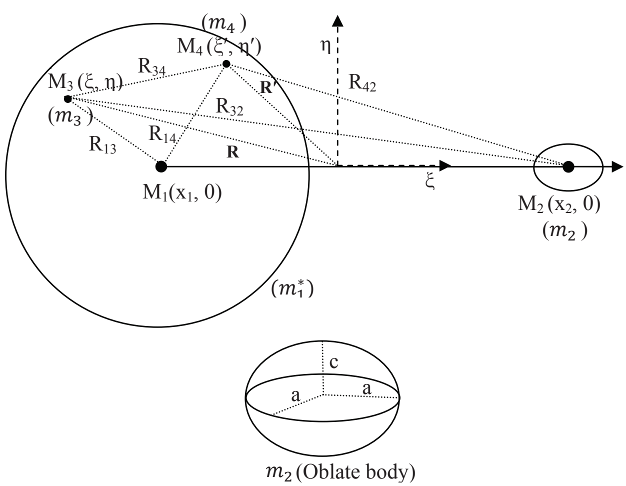

Figure 1. **Geometry of the Robe's restricted problem of $2+2$ bodies with the smaller primary m_2 an oblate body**

Various forces acting on M_3 with mass m_3 are:
1. The gravitational force F_{34} due to m_4

$$\mathbf{F_{34}} = \frac{Gm_3m_4\mathbf{R_{34}}}{R_{34}^3}.$$

2. The gravitational force F_A exerted by the fluid of density ρ_1

$$\mathbf{F_A} = -\left(\frac{4}{3}\right)\pi G\rho_1 m_3\mathbf{R_{13}},$$

where $\mathbf{R_{ij}} = \mathbf{M_iM_j}$, M_1 is the centre of the shell m_1^* and M_3 the centre of m_3.
3. The force of buoyancy F_B is

$$\mathbf{F_B} = \left(\frac{4}{3}\right)\pi\frac{G\rho_1^2 m_3\mathbf{R_{13}}}{\rho_3}.$$

4. The gravitational force $\mathbf{F_{32}}$ acting on m_3 due to the oblate body m_2 is

$$\mathbf{F_{32}} = Gm_2m_3\left(\frac{1}{R_{32}^3} + \frac{3}{2}\frac{A}{R_{32}^5}\right)\mathbf{R_{32}}$$

The equation of motion of m_3 in the inertial system is

$$m_3\ddot{\mathbf{R}} = \mathbf{F_{34}} + \mathbf{F_A} + \mathbf{F_B} + \mathbf{F_{32}}$$

$$\ddot{\mathbf{R}} = \frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} - \frac{4}{3}\pi G\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)\mathbf{R_{13}} + Gm_2\left(\frac{1}{R_{32}^3} + \frac{3}{2}\frac{A}{R_{32}^5}\right)\mathbf{R_{32}}$$

where $\mathbf{R} = \mathbf{OM_3}$ and $\mathbf{R_{ij}} = \mathbf{M_iM_j}$.
Now, we determine the equation of motion of m_3 in the synodic system.
In the rotating (synodic) system, the equation of motion of m_3 is

$$\begin{aligned}\frac{\partial^2\mathbf{r}}{\partial t^2} + 2\boldsymbol{\omega}\times\frac{\partial\mathbf{r}}{\partial t} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) =& \frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} - \frac{4}{3}\pi G\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)\mathbf{R_{13}} \\ &+ Gm_2\left(\frac{1}{R_{32}^3} + \frac{3}{2}\frac{A}{R_{32}^5}\right)\mathbf{R_{32}},\end{aligned} \tag{1}$$

where $\mathbf{r} = \mathbf{OM_3}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{k}} = $ (constant).
The angular velocity $\boldsymbol{\omega}$ of the oblate body m_2 is

$$\omega^2 = \frac{G\left(m_1^* + m_2\right)}{a'^3} + \frac{3}{10}\frac{G\left(m_1^* + m_2\right)\left(a^2 - c^2\right)}{a'^5}.$$

We, now, fix the units such that $m_1^* + m_2 = 1, a' = 1$. We choose t in such a way that $G = 1$.
We further take

$$\mu_1 = \frac{m_1^*}{m_1^* + m_2}, \mu_2 = \frac{m_2}{m_1^* + m_2} \text{ so that } \mu_1 + \mu_2 = 1.$$

Let $\mu_2 = \mu$, (say), then $\mu_1 = 1 - \mu$.
Thus the coordinates of m_1^* and m_2 are $(-\mu, 0), (1-\mu, 0)$.
In dimensionless variables, we have

$$\omega^2 = 1 + \frac{3}{2}A \text{ where } A = \frac{a^2 - c^2}{5a'^2}. \tag{2}$$

The Equations of motion of m_3 in the dimensionless cartesian coordinates are

$$\ddot{\xi} - 2\omega\dot{\eta} = V_\xi, \tag{3}$$

$$\ddot{\eta} + 2\omega\dot{\xi} = V_\eta, \tag{4}$$

where

$$V = \frac{\omega^2}{2}\left(\xi^2 + \eta^2\right) + \frac{\mu}{R_{32}} + \frac{\mu_4}{R_{34}} - \frac{K}{2}\left((\xi+\mu)^2 + \eta^2\right) + A\frac{\mu}{2R_{32}^3}$$

Similarly, in the rotating (synodic) system, the equations of motion of m_4 in dimensionless cartesian coordinates are

$$\ddot{\xi}' - 2\omega\dot{\eta}' = V'_{\xi'}, \tag{5}$$

$$\ddot{\eta}' + 2\omega\dot{\xi}' = V'_{\eta'}, \tag{6}$$

where

$$V' = \frac{\omega^2}{2}\left(\xi'^2 + \eta'^2\right) + \frac{\mu}{R_{42}} + \frac{\mu_3}{R_{43}} - \frac{K'}{2}\left((\xi'+\mu)^2 + \eta'^2\right) + A\frac{\mu}{2R_{42}^3}$$

and

$$\mu_3 = \frac{m_3}{m_1^* + m_2} \ll 1, K' = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_4}\right).$$

1. Equilibrium Solutions

The equilibrium solutions of m_3 and m_4 are given by

$$V_\xi = 0 = V_\eta \ ; \ V'_{\xi'} = 0 = V'_{\eta'}$$

i.e. ,

$$\xi - \mu_4\frac{(\xi - \xi')}{R_{34}^3} - K(\xi+\mu) - \mu\frac{(\xi - (1-\mu))}{R_{32}^3} + A\left(\frac{3}{2}\xi - \frac{3}{2}\mu\frac{(\xi-(1-\mu))}{R_{32}^5}\right) = 0, \tag{7}$$

$$\eta - \mu_4\frac{(\eta - \eta')}{R_{34}^3} - K\eta - \frac{\mu\eta}{R_{32}^3} + A\left(\frac{3}{2}\eta - \frac{3}{2}\frac{\mu\eta}{R_{32}^5}\right) = 0, \tag{8}$$

and

$$\xi' - \mu_3\frac{(\xi' - \xi)}{R_{43}^3} - K'(\xi'+\mu) - \mu\frac{(\xi' - (1-\mu))}{R_{42}^3} + A\left(\frac{3}{2}\xi' - \frac{3}{2}\mu\frac{(\xi'-(1-\mu))}{R_{42}^5}\right) = 0, \tag{9}$$

$$\eta' - \mu_3\frac{(\eta' - \eta)}{R_{43}^3} - K'\eta' - \frac{\mu\eta'}{R_{42}^3} + A\left(\frac{3}{2}\eta' - \frac{3}{2}\frac{\mu\eta'}{R_{42}^5}\right) = 0, \tag{10}$$

Case I: Collinear Equilibrium Solutions
By inspection, we see that the equations (8) and (10) are satisfied with $\eta = \eta' = 0$.

It remains to determine ξ and ξ' such that the following simplified forms of the Equations (7) and (9) are satisfied, i.e.

$$\xi - \mu\frac{\{\xi-(1-\mu)\}}{|\xi-(1-\mu)|^3} - \mu_4\frac{(\xi-\xi')}{|\xi-\xi'|^3} - K(\xi+\mu) + A\left[\frac{3}{2}\xi - \frac{3}{2}\frac{\mu\,(\xi-(1-\mu))}{|\xi-(1-\mu)\,|^5}\right] = 0 \quad (11)$$

$$\xi' - \mu\frac{\{\xi'-(1-\mu)\}}{|\xi'-(1-\mu)|^3} - \mu_3\frac{(\xi'-\xi)}{|\xi'-\xi|^3} - K'(\xi'+\mu) + A\left[\frac{3}{2}\xi' - \frac{3}{2}\frac{\mu\,(\xi'-(1-\mu))}{|\xi'-(1-\mu)\,|^5}\right] = 0 \quad (12)$$

In the Equation (11), when $\mu_4 = 0$ and $A = 0$ and in the Equation(12), when $\mu_3 = 0$ and $A = 0, (-\mu, 0)$ is the only equilibrium solution of the system ([**?**]). Now, we apply the perturbation theory when none of μ_3 , μ_4 ,A are zero.
We further define

$$\Omega(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{[\{x-(1-\mu)\}^2+y^2]^{\frac{1}{2}}} - \frac{K}{2}[(x+\mu)^2+y^2],$$

and

$$\Omega'(x,y) = \frac{1}{2}(x^2+y^2) + \frac{\mu}{[\{x-(1-\mu)\}^2+y^2]^{\frac{1}{2}}} - \frac{K'}{2}[(x+\mu)^2+y^2].$$

The solutions ξ and ξ' of (11) and (12) may be expressed as power series in small parameters ϵ_3 and ϵ_4 such that

$$\xi = -\mu + \sum_{j=1}^{\infty} a_{1j}\epsilon_4^j, \qquad \xi' = -\mu + \sum_{j=1}^{\infty} a_{2j}\epsilon_3^j. \quad (13)$$

where

$$\epsilon_i = \frac{\mu_i}{(\Lambda\mu_3+\mu_4)^{\frac{2}{3}}} \ll 1,\ \ (i=3,4)$$

and

$$\Lambda = \frac{l_1}{l_2} \quad (14)$$

$$l_1 = \Omega_{xx}^{(3)} + A\left(\frac{3}{2}+6\mu\right), l_2 = \Omega_{xx}^{('4)} + A\left(\frac{3}{2}+6\mu\right).$$

The upper suffix (3) and (4) denote the evaluation of the derivatives at the equilibrium solution $(-\mu, 0)$ for m_3 and m_4 respectively.
The Equations (7), (8), (9), and (10) can be written as

$$\Omega_x(\xi,\eta) - \mu_4\frac{(\xi-\xi')}{R_{34}^3} + A\left[\frac{3}{2}\xi - \frac{3}{2}\frac{\mu\,(\xi-(1-\mu))}{R_{32}^5}\right] = 0, \quad (15)$$

$$\Omega_y(\xi,\eta) - \mu_4\frac{(\eta-\eta')}{R_{34}^3} + A\left[\frac{3}{2}\eta - \frac{3}{2}\frac{\mu\eta}{R_{32}^5}\right] = 0, \quad (16)$$

$$\Omega'_x(\xi',\eta') - \mu_3\frac{(\xi'-\xi)}{R_{43}^3} + A\left[\frac{3}{2}\xi' - \frac{3}{2}\frac{\mu\,(\xi'-(1-\mu))}{R_{42}^5}\right] = 0, \quad (17)$$

$$\Omega'_y(\xi',\eta') - \mu_3\frac{(\eta'-\eta)}{R_{43}^3} + A\left[\frac{3}{2}\eta' - \frac{3}{2}\frac{\mu\eta'}{R_{42}^5}\right] = 0. \quad (18)$$

When $\eta = \eta' = 0$, the equilibrium solutions are given by the Equations

$$\Omega_x(\xi, 0) - \mu_4 \frac{(\xi - \xi')}{|\xi - \xi'|^3} + A\left[\frac{3}{2}\xi - \frac{3}{2}\frac{\mu\left(\xi - (1-\mu)\right)}{|\xi - (1-\mu)\,|^5}\right] = 0,$$
$$\Omega'_x(\xi', 0) - \mu_3 \frac{(\xi' - \xi)}{|\xi' - \xi|^3} + A\left[\frac{3}{2}\xi' - \frac{3}{2}\frac{\mu\left(\xi' - (1-\mu)\right)}{|\xi' - (1-\mu)\,|^5}\right] = 0.$$

Solving the two equations for ξ ,ξ' we get,

$$\xi = -\mu \pm \frac{\mu_4}{\left[(\mu_4 + \Lambda\mu_3)^2\left((1 - K + 2\mu) + \frac{3}{2}A + 6\mu A\right)\right]^{\frac{1}{3}}}$$

and

$$\xi' = -\mu \mp \frac{\mu_3\Lambda}{\left[(\mu_4 + \Lambda\mu_3)^2\left((1 - K + 2\mu) + \frac{3}{2}A + 6\mu A\right)\right]^{\frac{1}{3}}}$$

In the case when $\rho_3 = \rho_4$, we have
(i). $K = K'$
(ii). $\Omega(x, y) = \Omega'(x, y)$
(iii). $\Omega^{(3)}_{xx} = 1 - K + 2\mu = 1 - K' + 2\mu = \Omega^{'(4)}_{xx}$
(iv). $\Lambda = 1$
Therefore, ξ and ξ' become

$$\xi = -\mu \pm \frac{\mu_4}{[(\mu_3 + \mu_4)^2(1 - K + 2\mu + \frac{3}{2}A + 6\mu A)]^{\frac{1}{3}}},$$

and

$$\xi' = -\mu \mp \frac{\mu_3}{[(\mu_3 + \mu_4)^2(1 - K + 2\mu + \frac{3}{2}A + 6\mu A)]^{\frac{1}{3}}}.$$

Hence there are two equilibrium solutions of the system.

Case II: Non Collinear Equilibrium Solutions
In this case, we assume $\eta \neq 0,\ \eta' \neq 0$.
When $\mu_4 = 0$ and $A = 0$, the equilibrium solutions of m_3 lie on a circle with centre $(1-\mu, 0)$ and radius one, only when $K = 1 - \mu$, provided they lie within the spherical shell. Again when $\mu_3 = 0$ and $A = 0$, equilibrium solutions of m_4 lie on a circle with centre $(1-\mu, 0)$ and radius one, only when $K' = 1 - \mu$, provided they lie within the spherical shell([**?**]). Now, we apply the perturbation theory when none of μ_3 , μ_4 ,A are zero.
The equilibrium solutions of m_3 and m_4 are given by the Equations (7), (8), (9), and (10). The solutions of these equations may be expressed as power series in small parameters ϵ_3 and ϵ_4 such that

$$\xi = x' + \sum_{j=1}^{\infty} a_{1j}\epsilon_4^j, \qquad \eta = y' + \sum_{j=1}^{\infty} b_{1j}\epsilon_4^j, \tag{19}$$

$$\xi' = x'' + \sum_{j=1}^{\infty} a_{2j}\epsilon_3^j, \qquad \eta' = y'' + \sum_{j=1}^{\infty} b_{2j}\epsilon_3^j, \tag{20}$$

where

$$\epsilon_i = \frac{\mu_i}{(\Lambda\mu_3 + \mu_4)^{\frac{2}{3}}} \ll 1, \ \ (i = 3, 4) \tag{21}$$

and

$$\Lambda = \frac{l_1}{l_2},$$

$$l_1 = \Omega_{xx}^{(3)} + A\left(\frac{3}{2} + 6\mu\right), \quad l_2 = \Omega_{xx}^{\prime(4)} + A\left(\frac{3}{2} + 6\mu\right).$$

The upper suffix (3) and (4) denote the evaluation of the derivatives at the equilibrium solution (x', y') and (x'', y'') for m_3 and m_4 respectively, provided they lie within the spherical shell.

Here (x', y') is any point lying on the circle

$$\{(1 - \mu - x)^2 + y^2\} = 1$$

therefore, $x' = 1 - \mu - \cos(\phi), \ y' = \sin(\phi)$

where

$$180^\circ - \Theta \le \phi \le 180^\circ + \Theta \text{ and } \Theta = \sin^{-1}\frac{d}{2}$$

The portion of the circle with radius one which intersects m_1^* constitute the equilibrium solution of m_3. The highlighted portion in Figure 2 depicts this clearly.

Also, (x'', y'') is any point lying on the circle

$$\{(1 - \mu - x)^2 + y^2\} = 1$$

therefore, $x'' = 1 - \mu - \cos(\phi'), \ y'' = \sin(\phi')$, where $\phi \neq \phi'$ and

$$180^\circ - \Theta \le \phi' \le 180^\circ + \Theta$$

The non-collinear equilibrium solutions exist only when $K = 1 - \mu = K'$ and these inturn also imply $\rho_3 = \rho_4$. The Equations (7), (8), (9), and (10) can be written as (15), (16), (17), and (18).

To $o(\epsilon)$, where $\epsilon = \max(\epsilon_3, \epsilon_4)$ and using the values of $\xi, \ \eta, \ \xi', \ \eta'$ from the Equations (19) and (20) and applying Taylor's series, the first term of the Equation (15) is

$$\Omega_x(\xi, \eta) = a_{11}\epsilon_4\Omega_{xx}^{(3)} + b_{11}\epsilon_4\Omega_{xy}^{(3)} \tag{22}$$

We determine $\Omega_y(\xi, \eta)$, $\Omega'_x(\xi', \eta')$, and $\Omega'_y(\xi', \eta')$.

Since $\mu_3, \mu_4, \epsilon_3, \epsilon_4$ are very small, ignoring the higher order terms and using the Equations (19) and (20), the position of m_3 and m_4 are given by

$$\xi = x' + a_{11}\epsilon_4 = X_1 \, (say)\,. \tag{23}$$

$$\eta = y' + b_{11}\epsilon_4 = Y_1 \, (say)\,. \tag{24}$$

Also,

$$\xi' = x'' + a_{21}\epsilon_3 = X_2 \, (say)\,. \tag{25}$$

$$\eta' = y'' + b_{21}\epsilon_3 = Y_2 \, (say)\,. \tag{26}$$

We may note that

$$\begin{aligned}
\Omega^{(3)}_{xx} &= 3\mu \left(Cos^2\left(\phi\right)\right)\\
\Omega^{(3)}_{xy} &= -3\mu Cos\left(\phi\right) Sin\left(\phi\right)\\
\Omega^{(3)}_{yy} &= 1-\mu+3\mu \left(Sin^2\left(\phi\right)\right)\\
\Omega^{\prime(4)}_{xx} &= 3\mu \left(Cos^2\left(\phi^\prime\right)\right)\\
\Omega^{\prime(4)}_{xy} &= -3\mu Cos\left(\phi^\prime\right) Sin\left(\phi^\prime\right)\\
\Omega^{\prime(4)}_{yy} &= 1-\mu+3\mu \left(Sin^2\left(\phi^\prime\right)\right)
\end{aligned}$$

Equations (23), (24), (25) and (26) give approximate locations of the non-collinear equilibrium solutions. The equilibrium solutions of m_3 and m_4 is a perturbed curve of a circle. The portion of the curve which lies within the spherical shell constitute the set of equilibrium solutions. For the non-collinear equilibrium solutions to exist ρ_3 must be equal to ρ_4. It may be noted that there are an infinite number of non-collinear equilibrium solutions as ϕ, ϕ', $\phi \neq \phi'$ lie between the specified range. The positions of the non-collinear equilibrium solutions in this case are illustrated in Figure 3.

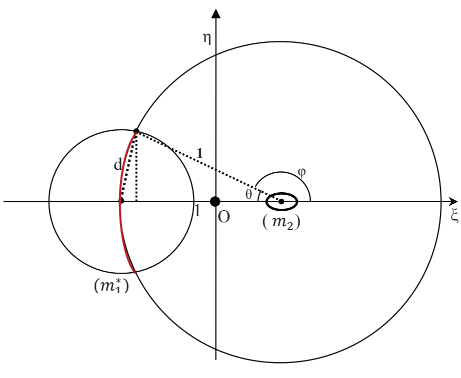

Figure 2. **Determination of the range of ϕ to evaluate the equilibrium solutions of m_3 (and similarly m_4)**

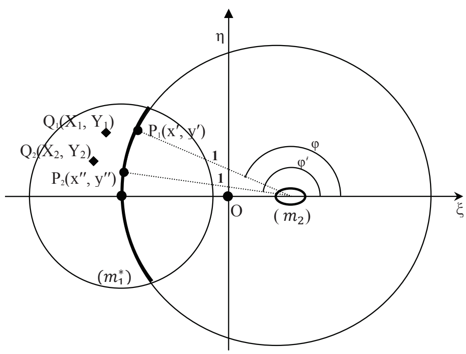

Figure 3. **Location of Non Collinear equilibrium solutions (when they exist) of the Robe's restricted problem of 2+2 bodies with the smaller primary $\mathbf{m_2}$ an oblate body**

2. Stability of the Collinear Equilibrium Solutions

The variational equations for m_3 are

$$\ddot{\alpha} - 2\omega\dot{\beta} = \alpha V_{\xi\xi}^{(3)} + \beta V_{\xi\eta}^{(3)}, \ddot{\beta} + 2\omega\dot{\alpha} = \alpha V_{\xi\eta}^{(3)} + \beta V_{\eta\eta}^{(3)}. \tag{27}$$

Now

$$V_{\xi\xi}^{(3)} = \omega^2 - K - 2\mu\left(1 + 3a_{11}\epsilon_4\right) + \frac{2\mu_4}{\mu_4 + \Lambda\mu_3}(1 - K + 2\mu) + A\left(\frac{3}{2} - 6\mu\left(1 + 5a_{11}\epsilon_4\right)\right) \tag{28}$$

$$V_{\xi\eta}^{(3)} = 0 \tag{29}$$

$$V_{\eta\eta}^{(3)} = \omega^2 - K + \mu\left(1 + 3a_{11}\epsilon_4\right) - \frac{\mu_4}{\mu_4 + \Lambda\mu_3}(1 - K + 2\mu) + A\left(\frac{3}{2} + \frac{3}{2}\mu\left(1 + 5a_{11}\epsilon_4\right)\right) \tag{30}$$

The variational equations become

$$\ddot{\alpha} - 2\omega\dot{\beta} = \alpha\left[(\omega^2 - K - 2A_1 + 2A_2) + A\left(\frac{3}{2} - 6A_3\right)\right] \tag{31}$$

$$\ddot{\beta} + 2\omega\dot{\alpha} = \beta\left[(\omega^2 - K + A_1 - A_2) + A\left(\frac{3}{2} + \frac{3}{2}A_3\right)\right] \tag{32}$$

where

$$A_1 = \mu\left(1 + 3a_{11}\epsilon_4\right)$$

$$A_2 = \frac{2\mu_4}{\mu_4 + \Lambda\mu_3}(1 - K + 2\mu)$$

$$A_3 = \mu\left(1 + 5a_{11}\epsilon_4\right)$$

The characteristic equation of m_3

$$\lambda^4 + p_1\lambda^2 + p_2 = 0 \tag{33}$$

where

$$p_1 = -\left(2\omega^2 - 2K - A_1 + A_2 + A\left(3 - \frac{9}{2}A_3\right) - 4\omega^2\right)$$

$$p_2 = \left[(\omega^2 - K - 2A_1 + 2A_2) + A\left(\frac{3}{2} - 6A_3\right)\right] \left[\left(\omega^2 - K + A_1 - A_2 + A\left(\frac{3}{2} + \frac{3}{2}A_3\right)\right)\right] \tag{34}$$

This is a quadratic equation in λ^2. Its roots are

$$\lambda^2 = \frac{(2\omega^2 - 2K - A_1 + A_2 + A\left(3 - \frac{9}{2}A_3\right) - 4\omega^2) \pm \sqrt{\Delta}}{2} \tag{35}$$

where

$$\Delta = \sqrt{p_1^2 - 4p_2} \tag{36}$$

The equilibrium point is stable if $p_1 > 0$,$p_2 > 0$, $\Delta > 0$.

Stability of Non Collinear Equilibrium Solutions The characteristic equation of m_3 (and similrly of m_4) is

$$\lambda^4 + p_1\lambda^2 + p_2 = 0 \tag{37}$$

where

$$p_1 = -\left(V_{\xi\xi}^{(3)} + V_{\eta\eta}^{(3)} - 4\omega^2\right) \tag{38}$$

$$p_2 = \left(V_{\xi\xi}^{(3)}V_{\eta\eta}^{(3)} - \left(V_{\xi\eta}^{(3)}\right)^2\right). \tag{39}$$

where the derivatives can be calculated as done in the previous case. The equilibrium point is stable if $p_1 > 0$,$p_2 > 0$, $\Delta > 0$ where $\Delta = \sqrt{p_1^2 - 4p_2}$.

Conclusion

Celestial bodies in general are not spherical, rather they are oblate or axis symmetric bodies. It is therefore essential that we concentrate on primaries which are axis symmetric bodies and preferably on oblate bodies. This paper is an extension to our paper [**?**]. In that paper, we had taken m_2 as mass point while in this paper we have considered m_2 as an oblate body. The non-collinear equilibrium solutions exist only when $\rho_3 = \rho_4$. There exist an infinite number of non collinear equilibrium solutions of the system, provided they lie inside the spherical shell (Figure 3). This problem has application in the motion of submarines in the earth moon setup. It observe that the problem of $2+2$ bodies can be extended to $2+n$ bodies taking $'n'$ infinitesimal masses within the ellipsoid. Earth is the only celestial body we know of which has fluid and the test particles within the fluid are considered as the submarines. In the solar or the extra solar system,we donot know of any other celestial body having liquid inside it. Our entire work is applicable in solar or extra solar system when and if such a system is discovered.

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Extension of Course Contents

A great deal of learning happens beyond the formal coursework. This section hence, aims to provide a creative, fertile setting for productive research that goes beyond the confines of classroom, and precincts of syllabi. It strengthens and expands the existing knowledge and adds interests to the course and provides an experience of transformative learning.

FERMAT'S LAST THEOREM

ANIKA JAIN

Abstract. Fermat's Last Theorem is the most notorious problem in the history of mathematics and surrounding it is one of the greatest stories imaginable. Pierre de Fermat created the Last Theorem while studying a page of Arithmetica which discussed various aspects of Pythagoras' Theorem. Fermat's Last Theorem was interesting yet challenging as it looked simple but was much harder than one can imagine to solve. The more that mathematicians tried, the more they failed, and the more desirable a solution became. The Last Theorem was solved in 1993 by Andrew Wiles which had defeated mathematicians for more than 300 years.

Introduction

The beauty of Fermat's Last Theorem lies in the fact that the problem itself is supremely simple to understand. It is a puzzle that is stated in terms familiar to every school child. However, the mathematics involved in its proof is some of the toughest in the world. Pierre De Fermat was born in 1601 in France. He was a high ranking judge. Though he was an amateur mathematician yet he was one of the most brilliant mathematicians who contirbuted to three significant areas of mathematics i.e. Calculus, Probability Theory and Number Theory.

There is no record of Fermat being inspired by a mathematical tutor, instead it was a copy of the Arithmetica written by Diophatus of Alexandria which became his tutor. He was the one who found the unique property of 26 which was the only number in the infinity of numbers to be sandwiched between a square and a cube i.e. $5^2 < 26 < 3^3$

Birth of the Riddle

One day, he was studying a particular page of Arithmetica which discussed the Pythagoras' Theorem which has infintely many solutions. Fermat must have been bored with this equation as it was too easy for him so he considered a slightly mutated version of the equation: $x^3 + y^3 = z^3$. He came to the conclusion that among the infinity of numbers there were none that fitted this new equation. Fermat claimed that the equation $x^n + y^n = z^n$ had no non trivial whole number solutions for n greater than 2. After the first marginal note that outlined the theory the mischievious genius jotted down an additional comment which would haunt generations of mathematicians.

> "I have a truly marvellous demonstration of this proposition
> which this margin is too narrow to contain."

Arithmeticorum Liber II. 61

interuallum numerorum 2. minor autem 1 N. atque ideo maior 1 N. + 2. Oportet itaque 4 N. + 4. triplos esse ad 2. & adhuc superaddere 10. Ter igitur 2. adscitis vnitatibus 10. æquatur 4 N. + 4. & fit 1 N. 3. Erit ergo minor 3. maior 5. & satisfaciunt quæstioni.

IN QVÆSTIONEM VII.

CONDITIONIS appositæ eadem ratio est quæ & appositæ præcedenti quæstioni, nil enim aliud requirit quàm vt quadratus interualli numerorum sit minor interuallo quadratorum, & Canones iidem hic etiam locum habebunt, vt manifestum est.

QVÆSTIO VIII.

PROPOSITVM quadratum diuidere in duos quadratos. Imperatum sit vt 16. diuidatur in duos quadratos. Ponatur primus 1 Q. Oportet igitur 16 − 1 Q. æquales esse quadrato. Fingo quadratum à numeris quotquot libuerit, cum defectu tot vnitatum quod continet latus ipsius 16. esto à 2 N. − 4. ipse igitur quadratus erit 4 Q. + 16. − 16 N. hæc æquabuntur vnitatibus 16 − 1 Q. Communis adiiciatur vtrimque defectus, & à similibus auferantur similia, fient 5 Q. æquales 16 N. & fit 1 N. 16/5 Erit igitur alter quadratorum 256/25. alter verò 144/25 & vtriusque summa est 400/25 seu 16. & vterque quadratus est.

OBSERVATIO DOMINI PETRI DE FERMAT.

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos & generaliter nullam in infinitum vltra quadratum potestatem in duos eiusdem nominis fas est diuidere cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

QVÆSTIO IX.

RVRSVS oporteat quadratum 16 diuidere in duos quadratos. Ponatur rursus primi latus 1 N. alterius verò quotcunque numerorum cum defectu tot vnitatum, quot constat latus diuidendi. Esto itaque 2 N. − 4. erunt quadrati, hic quidem 1 Q. ille verò 4 Q. + 16. − 16 N. Cæterum volo vtrumque simul æquari vnitatibus 16. Igitur 5 Q. + 16. − 16 N. æquatur vnitatibus 16. & fit 1 N. 16/5 erit

H iii

Figure 1

Then eventually Fermat died but his son Clement Sameul spent 5 years collecting his father's letters and notes and published his annotations in a special edition of the arithemetica called "Diophantus' Arithemetica containing observaton by P. De Fermat. Fermat's 48 observations reached the wider community, every single observation was eventually proved except Fermat's Last Theorem and since it was the last observation to be proved hence its name.

Developments to the solution

Leonhard Euler was one of the greatest mathematicians of the eighteenth century from Basle. He is famous for solving the question of 7 bridges of Konigsberg. He became frustrated by Fermat's Last Theorem and even asked a friend Clairaut to go back to Fermat's house to look for any scrap of paper that might give him a clue of Fermat's solution. Eventually he found a clue hidden in Fermat's jottings where he cryptically described the proof for the case n=4 using the method of Infinite Descent. This was the stepping stone for Euler and he thus proved Fermats Last Theorem for the case n=3.

Sophie Germain, a young French woman revolutionised the study of Fermat's Last Theorem. She worked on the Last Theorem for several years and eventually made a breakthrough. She adopted a new strategy to approach the problem. Her immediate goal was not to prove a particular case but to say something about many cases at once. Her calculation focused on particular prime p such that $2p + 1$ is also prime. Soon Peter Gustav Lejeune Dirichlet and Adrien-Marie Legendre proved the case n=5 based on Germain's observations. Few years later, Gabriel Lame proved Fermat's Last Theorem for the case n=7.

Elliptic curves and Modular Forms

Elliptic Curves are equations of the form: $y^2 = x^3 + ax^2 + bx + c$ where a, b, c are any whole numbers. The challenge is to find out if they have whole number solutions and, if so, how many. With an infinite quantity of whole numbers to investigate, giving a complete list of solutions to a particular equation is an impossible task. A simpler task is to look for solutions in a finite number space called clock arithmetic which involves truncating the line and looping it back on itself to form a number ring as opposed to a number line. For example, consider a 5- clock arithmetic where the number line has been truncated at 5 and looped back at 0. Consider the equation :

$$x^3 - x^2 = y^2 + y$$

The solutions are:

$$x = 0, y = 0$$
$$x = 0, y = 4$$
$$x = 1, y = 0$$
$$x = 1, y = 4$$

Although some of the solutions would not be valid in normal arithmetic, in 5 clock arithmetic they are acceptable. Mathematicians list the number of solutions in each clock arithmetic and call it the L- series for the elliptic equation.

Yutaka Taniyama and Goro Shimura were Japanese students studying at the University of Tokyo. Both were fascinated by an unfashionable topic called Modular Forms. It is a mathematical object that is symmetrical in an infinite number of ways which live in the upper half- plane of a four dimensional complex plane (x_r, x_i, y_r, y_i). The modular forms which live in hyperbolic space come in various shapes and sizes but each one is built from the same basic ingredient. The ingredients of modular form are labelled from one to infinity (M1, M2 ..). Every modular form has a modular series called the M- series, a list of ingredients and their quantity. The modular forms studied by Taniyama and Shimura can be shifted, switched, swapped, reflected and rotated in an infinite number of ways and still they remain unchanged.

They claimed that if elliptic curves over the field of rational numbers are related to modular forms through their respective L- series and M- series, the series were identical. This came to be known as the Taniyama-Shimura conjecture. It suggested deep fundamental relationship between two objects which came from opposite ends of mathematics.

During the autumn of 1984, a select group of number theorists gathered for a symposium in Oberwolfach, Germany where Gerhard Frey, one of the speakers made a remarkable claim that if anyone could prove the Taniyama-Shimura Conjecture, they would immediately prove Fermat‘s Last Theorem. Fermat's Last Theorem claims that there are no whole number solutions to the equation $x^n + y^n = z^n$, but Frey explored what would happen if it is false, i.e. that there is at least one solution. He labelled the unknown numbers with the letters A, B and C:

$$A^N + B^N = C^N$$

Frey then proceeded to 'rearrange' the equation. With a daft series of complicated manoeuvres, Frey fashioned Fermat's equation, with the hypothetical solution into a form similar to elliptic equations:

$$y^2 = x^3 + (A^N B^N)x^2 - A^N B^N$$

It is easier to appreciate the elliptical nature of Frey's equation if we let

$$a = A^N - B^N \ , \ b = A^N B^N \ , \ c = 0$$

Frey pointed out to his audience that this elliptic equation, created from the solution of Fermat's Last Theorem was so weird that the repercussions of its existence would be devastating for the Taniyama-Shimura conjecture. In fact it was so strange that it would be seemingly impossible for it to be related to a modular form. Ken Ribet, a professor of University of California at Berkeley, proved the Frey's claim. His colleague Barry Mazur was visiting Berkeley to attend the International Congress of Mathematicians. Ribet began explaining him a tentative strategy which he had been exploring when he realised that all he had to do was add some gamma-zero of (M) structure and just run through his argument. Thus, Ken Ribet with the help of Barry Mazur proved that the Taniyama-Shimura conjecture implies Fermat's Last Theorem.

"I think I'll stop here"

Andrew Wiles at the age of ten was fascinated by mathematics. As a child he loved to solve mathematical problems and riddles. In 1975 Andrew Wiles began his career as a graduate student at Cambridge University. He studied elliptic curves for his Ph.D. Thesis on the suggestion of John Coates, his mentor. For Andrew Wiles anything leading to Fermat's Last Theorem was worth pursuing. Coincidently whatever he did as a graduate student or professor developed ideas to prove the Taniyama-Shimura Conjecture. By fate, he had all the equipments to begin attack on the conjecture. For the next seven years he worked in complete secrecy trying to prove the conjecture.

After a year of contemplation, Wiles decided to adopt the strategy of *induction* to deal with infinity of numbers. He had to knock over a million dominoes. So instead he toppled the first domino and tried to show that it would lead to a domino-toppling effect. To prove the Taniyama-Shimura conjecture, mathematicians had to show that every one of the infinte number of elliptic equations could be paired with a modular form. Originally they tried to show that the whole E- series of one elliptic equation could be matched with the whole M-series of one modular form and then move onto the other. Instead of trying to match all elements of one E- series and M- series, Wiles tried to match one element of all E- series and M- series.He tried to show that the first gene in every E- series could be matched with the first gene of every M- series. Wiles used Galois's work on group theory as the foundation for his proof of the Taniyama-Shimura conjecture. A handful of solutions from every elliptic equation could be used to form a group.

For the next three years he worked on the Iwasawa theory but by the summer of 1991, he felt he had lost the battle to adapt it. Then he used the Kolyvagin-Flach method to extend his argument from the first piece of the elliptic equation to all pieces of the elliptic equation. Then around early January of 1993, he decided to confide in Nick Katz. Professor Nick Katz also worked in Princeton University's Mathematical Department and had known Wiles for several years. They decided that the best strategy would be to announce a series of lectures open to the department's graduate students. Wiles would give the course that would effectively cover the part of the proof that needed to be checked while Katz would be part of the audience along wth the graduates who had no idea of this. Once the lecture series was over Wiles devoted all his efforts to completing the proof.

After seven years of single-minded effort, Wiles had completed the proof of the Taniyama-Shimura conjecture and as a consequence had fulfilled his dream of proving Fermat's Last Theorem. On 23 June 1993, Andrew Wiles gave the third lecture at Isaac Newton Institute, Cambridge where he announced the proof. There was a typical dignified silence while he read out the proof and then wrote up the statement of Fermat's Last Theorem. He said, "I think I'll stop here", and then there as sustained silence. It was a historical event.

While the media circus continued and while mathematicians made the most of the spotlight, the serious work of checking was underway. Wiles submitted his manuscipt to the jounal *Inventiones Mathematicae*, whereupon its editor Barry Mazur began the process of selecting the referees. Unusually, the committee consisted six members. To simplify matters the 200-page proof was divided into six sections and each of the referees took responsibility for one of these chapters.

Chapter 3 was the responibility of Nick Katz, who had already examined that part of Wiles' proof earlier in the year. The proof was a gigantic argument, intricately constructed from hundreds of mathematical calculations glued together by thousand of logical links. One aspect of the argument did not make sense to Nick Katz. Sometime around 23 August he e-mailed Andrew who inturn sent him a fax explaining it but still Katz was not satisfied. He pointed out to an error in a crucial argument involving the Kolyvagin- Flach method which was supposed to extend the proof from the first element of all elliptic equations and modular forms to cover all the elements, providing the toppling mechanism from one domino to the next. The error was so abstract that it couldn't be described in simple works. The error did not necessarily mean that Wiles's work was beyond salvation, but it did mean that he would have to strengthen the argument.

He couldn't fix the mistake and hence went back to working in complete isolation. At this stage only the referees and Wiles knew about this error and things were going on in secrecy. Wiles himself didn't tell anything in order to buy some extra time. Months past and people began to wonder where the proof was that they were told about. Rumours of an error began to circulate. Eventually Wiles realised that he could not maintain his silence forever. After the autumn of dismal failure, he sent emails to the mathematical society telling them about an error in the proof.

Few were convinced by Wiles's optimism. Almost six months passed without the error being corrected. He decided to invite Richard Taylor, a Cambridge lecturer, to Princeton to work alongside him. Then in the spring of 1994, when it looked things could not get worse, an e-mail hit the computer screens stating, 'Noam Elkies has announced a counter example, so that Fermats Last Theorem is not true after all! The solution to Fermat that he constructs involves an incredibly larger exponent (larger than 10^{20}), but it is constructive. Noam Elkies back in 1988 had found a counter-example to Euler's conjecture, thereby proving it false. This was a tragic blow for Wiles- the reason he could not fix the proof was that the so called error as a direct result of the falsity of the Last Theorem. After one or to days of turmoil, mathematicians realised that the e-mail was dated 1 April. The e-mail was a mischievous hoax served as a suitable lesson for the Fermat rumour-mongers, and for a while the Last Theorem, Wiles, Taylor and the damaged proof were at peace.

They worked all summer but made no progress. Taylor had to go back to Cambridge to resume work so they kept September as a deadline. On 19 September, Wiles had a moment of revelation. During the seven years he had worked on two methods. Independently both the techniques were inadequate but the amalgamation of the Kolyvagin-Flach method and Iwasawa theory solved everything. Two papers consisting of 130 pages were published in *Annals of Mathematics* (May 1995). Thus Fermat's Last Theorem had been solved after 358 years.

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ROAD NETWORK ANALYSIS: A SHORTEST PATH ALGORITHM APPLICATION

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Abstract. A geographic information system (GIS) is a computer-based tool which can create, manipulate, analyze, store and display spatial information of objects and hence useful in transportation management. The shortest path problem is one of the classical problems of graph theory, which is the problem of finding a path between the source and the destination/target (analogous to vertices of graph) to minimize total cost or distance or time (analogous to edge-weight) between them. This paper compares Dijkstra algorithm and Bellman-Ford algorithm to find the shortest path. The purpose of the paper is to select one best algorithm from these two algorithms after conducting comparative analysis.

INTRODUCTION

Description. In graph theory, the shortest path problem is the problem of finding a path between two vertices (or nodes) in a graph such that the sum of the weights of its constituent edges is minimized. In terms of graph theory, let $e_{i,j}$ be the edge from vertex v_i to v_j. Given a directed graph *G=(V,E)* and a real-valued weight function, $f : E \to \mathbb{R}$, the goal is to find the shortest path from v_l to v_m, lets say $P_{l,m} = v_l \to v_1 \to v_2 \dots v_k \to v_m$ that minimizes the sum $f(e_{l,1}) + \sum_{i=1}^{k-1} f(e_{i,i+1}) + f(e_{k,m})\ \ \forall v_l, v_m \in V$.

Applications. It has many fundamental applications like Internet Routing (e.g. the OSPF routing algorithm), VLSI Routing, Traffic Information Systems, Robot Motion Planning, Routing Telephone Calls, Road Network Analysis, avoiding nuclear contamination, destabilizing currency marketone. We will focus on the application of shortest path algorithm on 'Road Network Analysis'.

SHORTEST PATH PROBLEM

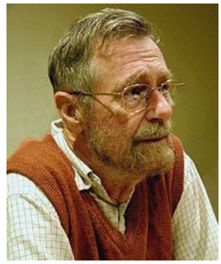
A road network can be considered as a positive-weighted digraph. (Self loops, if present, should be neglected in the graph.) The *nodes or vertices* represent road junctions (intersections) and each *edge* of the graph is associated with a road segment between two junctions. The *weight* of an edge may correspond to the length, time needed to transverse or the cost of the associated road segment.

Problem Definition: Given a graph $G = (V, E)$. Given any pair of vertices s and t, we would like to find a shortest path from source s to destination t i.e. a path from s to t in a graph G given by the sequence $s, v_1, v_2 \dots v_m, t$, such that the total weight of the edges $s \to v_1 \to v_2 \dots v_m \to t$ is minimum.

Though there are many shortest path algorithms like Dijkstra's, Bellman-Ford-Moore, Floyd-Marshal, Johnson's etc which can be used to solve the problem, we will focus on the working of the former two algorithms to solve the problem.

Dijkstra's Algorithm

Dijkstra's algorithm was conceived by computer scientist Edsger W. Dijkstra in 1956 while working at *Mathematical Center in Amsterdam* on a program to demonstrate capabilities of the new computer called *ARMAC*. After designing the shortest path algorithm, published in 1959, he implemented it for *ARMAC* for slightly simplified transportation map of 64 cities in Netherland (ARMAC was 6-bit computer and hence could hold 64 cities comfortably). Dijkstra's original variant found the shortest path between two nodes (all pairs shortest path problem), but a more common variant fixes a single node as the "source" node and finds shortest paths from the source to all other nodes in the graph, producing a shortest path tree (single-source shortest path problem) .



Edsger W. Dijkstra, (1930-2002)

Working of algorithm. Let the starting node be called the initial node. Let the distance of node v be the distance from the initial node to v. Dijkstra's algorithm will assign some initial distance values and will try to improve them step by step.

(1) Assign to every node a tentative distance value: set it to zero for our initial node and to infinity for all other nodes (this doesn't imply that there is an infinite distance, but it means that it's an unvisited node).
(2) Set the initial node as current. Mark all other nodes unvisited. Create a set of all the unvisited nodes called the unvisited set.
(3) For the current node, consider all of its unvisited neighbors and calculate their tentative distances. Compare the newly calculated tentative distance to the current assigned value and assign the smaller one.
(4) When we are done considering all of the neighbors of the current node, mark the current node as visited and remove it from the unvisited set. A visited node will never be checked again.
(5) If the destination node has been marked visited (when planning a route between two specific nodes) or if the smallest tentative distance among the nodes in the unvisited set is infinity (when planning a complete traversal; occurs when there is no connection between the initial node and remaining unvisited nodes), then stop. The algorithm is finished.
(6) Select the unvisited node that is marked with the smallest tentative distance, and set it as the new "current node" then go back to step 3.

Running Time. Bounds of the running time of Dijkstra's algorithm will be expressed using big-O notation. Dijkstra's algorithm can be implemented more efficiently by storing the graph in the form of adjacency lists and using a self-balancing binary search tree, binary heap, pairing heap, or Fibonacci heap (by Fredman & Tarjan, 1984) or Brodal queues in a

```
function Dijkstra(Graph, source):

    dist[source] ← 0                          // Distance from source to source
    prev[source] ← undefined                  // Previous node in optimal path initialization

    for each vertex v in Graph:  // Initialization
        if v ≠ source            // Where v has not yet been removed from Q (unvisited nodes)
            dist[v] ← infinity            // Unknown distance function from source to v
            prev[v] ← undefined           // Previous node in optimal path from source
        end if
        add v to Q                     // All nodes initially in Q (unvisited nodes)
    end for

    while Q is not empty:
        u ← vertex in Q with min dist[u]  // Source node in first case
        remove u from Q

        for each neighbor v of u:          // where v is still in Q.
            alt ← dist[u] + length(u, v)
            if alt < dist[v]:              // A shorter path to v has been found
                dist[v] ← alt
                prev[v] ← u
            end if
        end for
    end while

    return dist[], prev[]

end function
```

```
function Dijkstra(Graph, source):
    dist[source] ← 0                        // Initialization
    for each vertex v in Graph:
        if v ≠ source
            dist[v] ← infinity              // Unknown distance from source to v
            prev[v] ← undefined             // Predecessor of v
        end if
        Q.add_with_priority(v, dist[v])
    end for

    while Q is not empty:                 // The main loop
        u ← Q.extract_min()               // Remove and return best vertex
        for each neighbor v of u:
            alt = dist[u] + length(u, v)
            if alt < dist[v]
                dist[v] ← alt
                prev[v] ← u
                Q.decrease_priority(v, alt)
            end if
        end for
    end while
    return prev[]
```

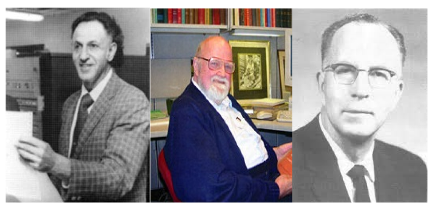
FIGURE 1. Djikstra's Algorithm using Simple linked list or array and using Minimum priority queue

priority queue to implement extracting minimum efficiently than storing the vertex set Q as an ordinary linked list or array in Simple Dijkstra's algorithm since *extract-minimum* is simply a linear search through all vertices in Q. The running time for Simple Dijkstra's is $O(|E| + |V|^2) = O(|V|^2)$ (for sparse graphs, its even fewer).The Dijkstra's algorithm using minimum priority queue takes $O(|E| \cdot \log(|V|)$ in case of self-balancing binary search tree or ordinary binary heap (for connected graphs) and takes $O(|E| + |V| \cdot \log(|V|)$ in case of the Fibonacci heap. The average case time complexity is lower than the worst-case (assume edge-costs are drawn independently from common probability distribution) . Since the expected number of *decrease-key operations*, in this case is bounded by $O\left(|V| \cdot \log\left(\frac{|E|}{|V|}\right)\right)$, giving total running time of $O\left(|E| + |V| \cdot \log\left(\frac{|E|}{|V|}\right)\right)$.

Bellman-Ford-Moore Algorithm

The Bellman-Ford algorithm is Single Source Shortest Path Algorithm for weighted digraph.

The algorithm is usually named after two of its developers, Richard Bellman and Lester Ford Jr., who published it in 1958 and 1956 respectively. However Edward F. Moore also published the same algorithm in 1957, and hence, sometimes called the Bellman-Ford-Moore algorithm.



Bellman-Ford-Moore

Working of Algorithm. The algorithm initializes the distance to the source as 0 and all other nodes to ∞. Then for all edges, if the distance to the destination can be shortened by taking the edge, the distance is updated to the new lower value. At each iteration that the edges are scanned, the algorithm finds all shortest paths of at most length i edges. Since the longest possible path without a cycle can be $|V| - 1$ edges, the edges must be scanned $|V| - 1$ times to ensure the shortest path has been found for all nodes. A final scan of all the edges is performed and if any distance is updated, then a path of length $|V|$ edges has been found which can only occur if at least one negative cycle exists in the graph and return it's value (if present) and also return shortest path. This algorithm finds shortest path in bottom up manner.

```
function BellmanFord(list vertices, list edges, vertex source)
   ::distance[],predecessor[]

   // This implementation takes in a graph, represented as
   // lists of vertices and edges, and fills two arrays
   // (distance and predecessor) with shortest-path
   // (less cost/distance/metric) information

   // Step 1: initialize graph
   for each vertex v in vertices:
       if v is source then distance[v] := 0
       else distance[v] := inf
       predecessor[v] := null

   // Step 2: relax edges repeatedly
   for i from 1 to size(vertices)-1:
       for each edge (u, v) with weight w in edges:
           if distance[u] + w < distance[v]:
               distance[v] := distance[u] + w
               predecessor[v] := u

   // Step 3: check for negative-weight cycles
   for each edge (u, v) with weight w in edges:
       if distance[u] + w < distance[v]:
           error "Graph contains a negative-weight cycle"
   return distance[], predecessor[]
```

FIGURE 2. Bellman-Ford Algorithm pseudo-code

Running Time. Bellman-Ford Algorithm runs in $O(|V| \cdot |E|)$ time.

COMPARISON OF DIJKSTRA AND BELLMAN-FORD ALGORITHM

Similarities.

- **Aim**: Both algorithms are used to find the shortest path of a network where the value of the shortest path may vary according to the situation of the (road) network by solving cost and length problems.
- **Principle of relaxation**: Both algorithms work on relaxation principle which is an approximation to the correct distance found by gradually replacing it by more accurate values until the optimum solution is reached i.e. approximate distance to each vertex is always an overestimate of the true distance, and is replaced by the minimum of its old value and getting the length of a newly found path.

Dissimilarities.

- **Time Complexity**: The Simple Dijkstra algorithm takes $O(|V|^2)$ time (even lesser in algorithm with minimum priority queue) whereas by Bellman Ford algorithm is $O(|V|.|E|)$, and hence had more running time than Dijkstra's.
- **Space Complexity**: Implementing Q using priority queue with at most $|E|$ edges in the heap, the space complexity of Dijkstra algorithm is $O(|V| + |E|)$ and of Bellman Ford algorithm is $O(|V|)$.So Bellman Ford algorithm takes more space than Dijkstra's and hence not suggested for larger networks (specifically for GIS, which involves analysis of lots of data) and for dense networks.

- **Negative weighted and negative cycled graphs**: Unlike Dijkstra algorithm, Bellman-Ford algorithm works in negative weighted graph and also do not detect negative cycle (but in GIS, distance is in positive form, hence not an added advantage) .
- **Count to infinity**: In Bellman Ford algorithm if link or node failures render a node unreachable from some set of other nodes, those nodes may spend forever gradually increasing their estimates of the distance to it, and in the meantime there may be routing loops. This slowly propagates through the network until it reaches infinity (in which case the algorithm corrects itself, due to the "Relax property" of Bellman Ford).
- **Update of information**: Unlike Dijkstra Algorithm, in Bellman Ford algorithm changes in network topology are not reflected quickly, since updates are spread node-by-node. It does not scale well, but in GIS, for finding shortest path, we have to update the traffic information quickly.
- **Difference of working of relaxation principle**: Dijkstra algorithm greedily selects the minimum-weight node that has not yet been processed, and performs this relaxation process on all of its outgoing edges whereas the Bellman-Ford algorithm simply relaxes all the edges in $|V| - 1$ times. In each of these repetitions, the number of vertices with correctly calculated distances grows gradually to all the vertices. This method allows the Bellman-Ford algorithm to be applied to a wider class of inputs than Dijkstra.
- **Versatility**: The Djikstra algorithm uses a priority queue data structure which can be implemented in a number of different ways like binary heap or fibonacci heap.The latter is smaller for fairly dense graphs (i.e. graphs where $V = |E|$).
- **Implementation**: Although the proof of correctness is a bit technical, the Bellman-Ford algorithm is easy to implement and doesn't use any complicated data structures whereas Fibonacci heaps are difficult to implement and have poor constant factors.

CONCLUSION

After comparative analysis of both the algorithms, it can be concluded that Dijkstra's algorithm is more efficient, faster and updates information quickly than Bellman Ford algorithm for road network analysis. It also works better and faster in larger network. Hence Dijkstra's algorithm is preferred over Bellman-Ford algorithm for road network analysis in GIS technology and is widely used in real time application in GIS.

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PERTURBATION TECHNIQUES

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Abstract. We come across various equations arising from mathematical models which cannot be solved exactly. Thus we must resort to approximate and numerical methods. The main emphasis of the paper is to solve various differential and polynomial equations which have been perturbed due to the presence of a small parameter say $0<\epsilon<<1$. This paper will focus on regular and singular perturbation techniques to obtain approximate solution of such problems.

Introduction

Consider a problem:

$$P^{\epsilon}(x)=0 \tag{1}$$

depending on a small, real-valued parameter ϵ that simplifes in some way when ϵ (for example, it is linear or exactly solvable). The aim of perturbation theory is to determine the behavior of the solution x = x^{ϵ} of (1) as $\epsilon \to 0$.

In this paper, we will discuss two types of perturbations: regular and singular perturbation.

Regular Perturbation

A regular perturbation problem is one for which the perturbed problem for small, nonzero values of ϵ is qualitatively the same as the unperturbed problem for $\epsilon= 0$. One typically obtains a convergent expansion of the solution with respect to ϵ, consisting of the unperturbed solution and higher order corrections. The basis of regular perturbation method is to assume a solution of the problem in terms of a power series in ϵ of the form

$$y_0(t)+\epsilon y_1(t)+\epsilon^2 y_2(t)+....$$

The simplest problem which can be addressed by regular perturbation theory is that of finding the roots of polynomials.

Example. Consider the following cubic equation:

$$x^3-x+\epsilon=0, \qquad 0<\epsilon<<1 \tag{2}$$

When $\epsilon \to 0$ in the above equation,we obtain 0,±1 as the solution.

For regular perturbation, we consider x as a taylor-like expansion in the parameter ϵ

$$x=x_0+x_1\epsilon+x_2\epsilon^2+x_3\epsilon^3.....$$

Substituting this is the cubic equation and collecting terms in power of ϵ, we get:

$$(x_0^3 - x_0) + (3x_0^2 x_1 - x_1 + 1)\epsilon + (3x_0 x_2 - x_2 + 3x_0 x_1^2)\epsilon^2 + ... = 0$$

This gives

$$x_0 = 0, \pm 1, x_1 = \frac{1}{1 - 3x_0^2}$$

Thus the corresponding solutions of the equation (2) are:

$$x = \epsilon + O(\epsilon^2)$$

and,

$$x = \pm 1 - \frac{1}{2}\epsilon + O(\epsilon^2)$$

Note that as ϵ=0 in the above solutions, we get x=0,$\pm$1, which coincided with the exact solutions of cubic eqaution when ϵ=0

Now we discuss an application of the regular perturbation technique.

Motion in a Nonlinear Resistive Medium:

Consider a body of mass m with initial velocity V_0 moves in a medium that offers a resistive force of maginitude av-bv^2 where v=v(τ) is the velocity of the object as a function of time τ and a and b are positive constants b<<a . Thus we see that since b <<a, the non linear part (v^2) of the force is assumed be small as compared to the linear part (v).

By Newton's second law, the equation of motion for the system is :

$$m\frac{dv}{d\tau} = -av + bv^2, \quad v(0) = V_0 \tag{3}$$

Now, to solve this equation using regular perturbation we convert equation (3) into dimensionless variable. Let

$$y = \frac{v}{V_0}, \qquad t = \frac{\tau}{(m/a)}$$

Thus we have

$$\frac{dy}{dt} = \frac{1}{V_0}\frac{dv}{dt} = \frac{1}{V_0}\frac{dv}{d\tau}\frac{d\tau}{dt} = \frac{m}{aV_0}\frac{dv}{d\tau}$$

Substituting this in equation (3), we get:

$$\frac{dv}{dt} = -y + \epsilon y^2, \quad t > 0, \quad y(0) = 1 \tag{4}$$

where

$$\epsilon = \frac{bV_0}{a}$$

Note that ϵ <<1 since b<<a

Equation (4) is a slightly perturbed form of the linear equation

$$\frac{dy}{dt} = -y, \quad y(0) = 1$$

which is easily solved by

$$y = \exp(-t) \tag{5}$$

Now we will check if this is a good approximation to the solution of (4) by regular perturbation

Let

$$y = y_0(t) + y_1(t)\epsilon + y_2(t)\epsilon^2 + y_3(t)\epsilon^3..... \tag{6}$$

We determine y_0,y_1,y_2,... by substituting (6) in (4),

The differential equation becomes

$$\frac{dy_0}{dt} + \epsilon\frac{dy_1}{dt} + \epsilon^2\frac{dy_2}{dt} + ... = -(y_0 + \epsilon y_1 + \epsilon^2 y_2 + ...) + \epsilon(y_0 + \epsilon y_1 + \epsilon^2 y_2 + ...)^2$$

By comparing the coefficients on both sides we get:

$$\frac{dy_0}{dt} = -y_0,$$
$$\frac{dy_1}{dt} = -y_1 + y_0^2,$$
$$\frac{dy_2}{dt} = -y_2 + 2y_0y_1, ...$$

Since y(0)=1, we have

$$y_0(0) + \epsilon y_1(0) + \epsilon^2 y_2(0) + ... = 1$$

i.e

$$y_0(0) = 1, \;\; y_1(0) = y_2(0)... = 0$$

The above differential equations can be solved exactly:

$$y_0 = e^{-t}$$
$$y_1 = e^{-t} - e^{-2t}$$
$$y_2 = e^{-t} - 2e^{-t} + e^{-3t}, ..$$

Thus the perturbation solution is

$$y = \exp(-t) + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + ...$$

Note that as $\epsilon \to 0$ in the above equation, the solution coincides with equation (5).

Now let us consider a situation where regular perturbation fails:

Example. Consider the boundary problem

$$\epsilon\frac{d^2y}{d\tau^2} + (1+\epsilon)\frac{dy}{d\tau} + y = 0, 0 < t < 1, 0 < \epsilon << 1 \;\; y(0) = 0, y(1) = 0 \tag{7}$$

Let

$$y = y_0(t) + y_1(t)\epsilon + y_2(t)\epsilon^2 + y_3(t)\epsilon^3... \tag{8}$$

Substitution into differential equation (7) gives :

$$\epsilon\left(\frac{d^2y_0}{d\tau^2} + \epsilon\frac{d^2y_1}{d\tau^2} + \epsilon^2\frac{d^2y}{d\tau^2} + ..\right) + (1+\epsilon)\left(\frac{dy_0}{d\tau} + \epsilon\frac{dy_1}{d\tau} + \epsilon^2\frac{dy_2}{d\tau}\right) + (y_0 + \epsilon y_1 + \epsilon^2 y_2 + ..) = 0$$

and equating to zero the coefficients of power of ϵ gives :

$$\frac{dy_0}{d\tau} + y_0 = 0$$

$$\frac{dy_1}{d\tau} + y_1 = -\frac{d^2 y_0}{d\tau^2} - \frac{dy_0}{d\tau} \dots$$

Since y(0)=0 and y(1)=1, we have:

$$y_0(0) = 0, y_0(1) = 1$$

But a first order differential equation has two conditions to satisfy, thus $y_0(0)=0$ gives $y_0(t)=0$ and $y_0(1)=1$ gives $y_0(t)=\exp(1-t)$. This function cannot satisfy the condition at t=0.Therefore we are at an impasse, regular perturbation has failed at the first step.

There are many instances when regular perturbation fails. Following are the indicators that suggest its failure:

- When the small parameter say ϵ is multiplied with the highest derevative of the differential equation.
- When setting the parameter to zero completely changes the character of the problem.
- When singular points are present in the interval of interest.
- When problem occurs on infinite domains.

In the above boundary value problem the order of the ordinary differential equation reduces from two to one when ϵ=0. We solve such problems by singular perturbation method.

Singular Perturbation

A singular perturbation problem is one for which the perturbed problem is qualitatively different from the unperturbed problem. We solve the differential equation (7) using the following steps:

(1) **Exact solution**:The above differential equation can be solved exactly since is a linear equation with constant coefficients and the solution is given by:

$$y(t) = \frac{e^{-t} - e^{-\frac{t}{\epsilon}}}{e^{-1} - e^{-\frac{1}{\epsilon}}}$$

Easy calculation shows that

$$\frac{d^2y}{dt^2}(0) = O(\epsilon^{-2})$$

and

$$\frac{d^2y}{dt^2}(0.5) = O(1)$$

Hence $\ddot{y}$ is very large when $\epsilon \to 0$, and therefore the term $\epsilon\ddot{y}$ is not small as would be anticipated in a regular perturbation calcululation. This region near the origin is known as the inner region and the region away from the origin where the term $\epsilon\ddot{y}$ is small and may be safely neglected is known as the outer region.

Thus by setting ϵ=0 in the original problem is a valid approximation provided we take only the right(t=1) boundary condition. From above we see that $y_0(t)=\exp(1-t)$. This approximate solution which is valid in the outer region is known as the outer approximation.

Definition: The region near t=0 where y is changing rapidly is called the boundary layer.

(2) **Balancing**: To analyse the behaviour in the boundary layer, we notice that significant changes in y takes place in a very short time, which suggests a time scale on the order of some function of ϵ, say δ (ϵ)

Let

$$t = \frac{\tau}{\delta(\epsilon)}, \quad Y(t) = y(\tau)$$

After substitution, the diffential equation (7) becomes

$$\frac{\epsilon}{\delta(\epsilon)^2}\frac{d^2Y}{dt^2} + \frac{(1+\epsilon)}{\delta(\epsilon)}\frac{dY}{dt} + Y = 0 \tag{9}$$

Now if the equation has been correctly rescaled for t in the boundary layer, then Y and its derevatives should be O(1), with the magnitudes of the terms given by the coefficients:

$$\frac{\epsilon}{\delta(\epsilon)^2}, \quad \frac{1}{\delta(\epsilon)}, \quad \frac{\epsilon}{\delta(\epsilon)}, \quad 1$$

We determine $\delta(\epsilon)$ by seeking a two term dominant balance that will allow us to simplify equation (9). The simplified equation should yield an approximation to Y(t) that satisfies the boundary condition at t = 0 as well as a matching condition at the right edge of the boundary layer. If it is to meet both requirements, the approximation must be the solution to a second-order equation. For this reason, one of the terms in the dominant balance must be $\frac{\epsilon}{\delta(\epsilon)^2}$. So we have the following cases:

- $\frac{\epsilon}{\delta(\epsilon)^2}$ and $\frac{1}{\delta(\epsilon)}$ are dominant. This yields $\delta(\epsilon)$=O(ϵ), so that the dominant terms are O(ϵ^{-1}) and the others O(1).
- $\frac{\epsilon}{\delta(\epsilon)^2}$ and 1 are dominant. Thus $\delta(\epsilon)$=O($\sqrt{\epsilon}$) so that the dominant terms are O(1) and the neglible ones O($\frac{1}{\sqrt{\epsilon}}$). Clearly (b) won't work.
- $\frac{\epsilon}{\delta(\epsilon)^2}$ and $\frac{\epsilon}{\delta(\epsilon)}$ are dominant. Thus $\delta(\epsilon)$= O(1), which gives dominant terms of O(ϵ) and negligible ones of O(1). This clearly won't work either.

(3) **Inner approximation**: Since only the first part above is possible we take $\delta(\epsilon)$=ϵ. Substituting this in (9) we get

$$\frac{d^2Y}{dt^2} + \frac{dY}{dt} + \epsilon\frac{dY}{dt} + \epsilon Y = 0 \quad Y(0) = 0 \tag{10}$$

Since the scaling is now assumed to be correct, we may resort to regular perturbation. So let

$$Y = Y_0(t) + Y_1(t)\epsilon + Y_2(t)\epsilon^2 + Y_3(t)\epsilon^3... \tag{11}$$

and substitution of (11) in (10) gives:

$$(\frac{d^2Y_0}{dt^2} + \frac{d^2Y_1}{dt^2}\epsilon + \frac{d^2Y_2}{dt^2}\epsilon^2 + ...) + (1+\epsilon)(\frac{dY_0}{dt} + \frac{dY_1}{dt}\epsilon + ..) + \epsilon(Y_0 + Y_1\epsilon + ...)$$

This gives the leading order problem
$\ddot{Y}_0+\dot{Y}_0=0$ for t=O(1) , $Y_0(0)=0$
The solution of the above system is:
$Y_0(t)=A(1-\exp(-t))$
Thus the inner approximation is given by: $y_i(\tau)=A(1-\exp(\frac{-\tau}{\epsilon}))$

(4) **Matching**: In order to find the constant A, it seems reasonable that the inner and outer expansions should agree to some order in an overlap domain that is intermediate between the boundary layer and outer region. If t=O(ϵ), then t is in the boundary layer and if t=O(1), then t is in the outer region; therefore, this overlap domain could be characterized as values of t for which t=O($\sqrt{\epsilon}$), for example,since orderwise $\sqrt{\epsilon}$ is between ϵ and 1. This is because $\sqrt{\epsilon}$ goes to zero slower than ϵ does. This intermediate scale alows the introduction of a new scaled independent variable η in the overlap domain defined by:

$$\eta = \frac{\tau}{\sqrt{\epsilon}}$$

For matching we require that the inner approximation written in terms of the intermediate variable η should agree in the limit as $\epsilon \to 0^+$ with the outer approximation written in terms of the intermediate variable. In symbols for matching we require that for fixed η

$$\lim_{\epsilon\to 0^+} y_0(\sqrt{\epsilon}\eta) = \lim_{\epsilon\to 0^+} y_i(\sqrt{\epsilon}\eta)$$

In our problem

$$\lim_{\epsilon\to 0^+} y_0(\sqrt{\epsilon}\eta) = \lim_{\epsilon\to 0^+} \exp(1-\sqrt{\epsilon}\eta) = e$$

and

$$\lim_{\epsilon\to 0^+} y_i(\sqrt{\epsilon}\eta) = \lim_{\epsilon\to 0^+} A\Big\{1-\exp(\frac{-\eta}{\sqrt{\epsilon}})\Big\} = A$$

Thus matching requires that A=e and the final inner approximation solution is:
$y_i(\tau)=e(1-e^{\frac{-\tau}{\epsilon}})$ for $\tau=O(\epsilon)$

(5) **Uniform approximation:** To obtain an approximation y_u that is valid uniformly on [0, 1], we add the inner and outer appoximations and subtract their common limit obtained in matching in the intermediate zone:

$$y_u(\tau) = y_i(\tau) + y_0(\tau) - e = e^{1-t} - e^{(1-\frac{t}{\epsilon})}$$

Thus $y_u(\tau)$ provides a uniform approximate solution throughout the interval [0,1]. Substituting $y_u(\tau)$ into the differential equation gives

$$\epsilon\frac{d^2y_u}{d\tau^2} + (1+\epsilon)\frac{dy_u}{d\tau} + y_u = 0$$

So $y_u(\tau)$ satisfies the differential equation exactly on (0,1). Checking the boundary conditions
$y_u(0)=0$ and $y_u(1)=1-e^{1-\frac{1}{\epsilon}}$

The left boundary condition is satisfied exactly and the right boundary condition holds for O(ϵ^n) for any n>0 since

$$\lim_{\epsilon \to 0^+} \frac{e^{1-\frac{1}{\epsilon}}}{\epsilon^n} for\ any\ n > 0$$

We conclude this paper with a general singular perturbation theorem for linear equations with variable coefficients. The proof of the theorem will follow exactly as we solved the above example.

Theorem: Let p(t) and q(t) be continuous, with p(t) > 0 on [0, 1]. For the boundary value problem

$$\epsilon \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, for 0 < t < 1, 0 << 1$$

y(0) = a, y(1) = b there exists a boundary layer at t = 0 with inner and outer approximations given respectively by

$$y_i(t) = A + (a - A)\exp\left(\frac{-p(0)t}{\epsilon}\right)$$

$$y_0(t) = b \exp \int_t^1 \frac{q(s)\,ds}{p(s)}$$

where

$$A = b \exp \int_0^1 \frac{q(s)\,ds}{p(s)}$$

CONCLUSION

Perturbation theory first appeared in one of the oldest branches of applied mathematics: celestial mechanics, the study of motion of the planets. The scope of the perturbation theory at the present time is much broader than its applications to celestial mechanics but the main idea is the same. One of the major applications of Perturbation techniques is the WKB method (named after the scientists Wentzel-Kramers-Brillouin) that applies to a variety of problems, in particular to linear differential equations. Also it can be used to determine the large eigenvalues for simple differential operators.

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Interdisciplinary Aspects of Mathematics

Mathematics is just not a classroom discipline but a tool for organizing and understanding various concepts and applications. This section covers topics that delve into other disciplines, integrating the mode of thinking and knowledge of the respective discipline with Mathematics. The section hence highlights the cosmic scope of Mathematics, leveraging its amalgamation with other disciplines.

GOOGLE PAGE RANK ALGORITHM

YASHASWIKA GAUR

Abstract. When Google went online in the 90s, nobody knew that it would become the most popular search engine in the world. One of the reasons why Google is such an effective search engine is the PageRank algorithm developed by its founders, Larry Page and Sergey Brin, when they were graduate students at Stanford University. PageRank is determined entirely by the link structure of the World Wide Web. It is an automated process that determines the relevance order of the search items on the web. This helps Google organise all the data on the internet and provide the most useful pages on the top, hence making the user's search easier and productive. This paper will describe the theorem and how it works.

Introduction

The usefulness of a search engine depends on the relevance of the results it gives. What makes Google the most widely used search engine is the page rank algorithm behind it, which quantitatively rates the importance of each page on the web, allowing Google to rank the pages and therefore present the most relevant and helpful pages first. The idea that Page Rank brought up was that, the importance of any web page can be judged by looking at the pages that link to it. If we create a web page i and include a hyperlink to the web page j, this means that we consider j important and relevant for our topic. If there are a lot of pages that link to j, this means that the common belief is that page j is important. If on the other hand, j has only one backlink, but that comes from an authoritative site k, we say that k transfers its authority to j; in other words, k asserts that j is important. Whether we talk about popularity or authority, we can iteratively assign a rank to each web page, based on the ranks of the pages that point to it.

What is page rank?

PageRank is very simply a 'vote' by all other pages on the web about how important a particular page is. A link to a page counts as a vote of support. PageRank can be calculated using a simple iterative algorithm and corresponds to the principal eigenvector of the normalised link matrix of the web. Also, the importance score or just score refers to any quantitative rating of the importance of a webpage. This will always be a non negative number. The core idea in assigning a score to any given web page is that the page's score is derived from the links made to that page from other web pages. Suppose the web consists of n pages, each page indexed by an integer k where k lies between 1 and n. A graph in which the arrows from a Page A to Page B indicate the link from Page A to Page B is called a directed graph. We will begin by picturing the web as a directed graph with nodes represented by web pages and the edges represented by the links between them. Suppose we have an internet consisting of four pages to start with, say Page 1, Page 2, Page 3, Page 4. Let the pages be connected in such a way that Page 1 hyperlinks to pages 2,3 and 4; Page 2 to pages 3 and 4; Page 3 to page 1; page 4 connects to pages 3 and 1. We make the directed graph with four nodes respectively one for each site. When the page i references j, we add a directed edge between node i and node j in the graph. For the purpose of calculating the PageRank, we will ignore navigational links and focus only on the connections between different websites. For instance, Page 1 links to all the other pages, so Node 1 in the graph will have outgoing edges to all the other nodes. After analyzing each web page, we get the graph in Figure 1.

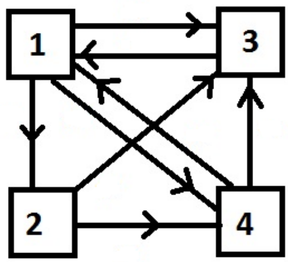

Figure 1

In our model, each page should transfer its importance evenly to the pages that it links to. Node 3 has three outgoing edges, so it will pass on 1/ 3 of its importance score to each of the other three nodes. In general if a node has k outgoing edges, it will pass on $1/k$ of its importance to each of the nodes that it links to. Hence, we get the graph in Figure 2 after assigning weights to each edge.

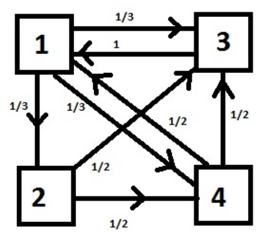

Figure 2

We construct a column stochastic matrix of the above graph, A $= \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$

Suppose that initially the importance is uniformly distributed among the four nodes, each getting 1/4 part of the total score of that page . Denote by v, the initial rank vector, having all entries equal to 1/4. Each incoming link increases the importance of a web page, so we update the rank of each page by adding to the current value the importance of the incoming links. This is same as multiplying the matrix A with v. At step 1, the new importance vector is $v_1= Av$. We thus iterate the process and consequently find $v_2= A(Av)$. Numeric computation gives:

$$\text{for v}=\begin{bmatrix}0.25\\0.25\\0.25\\0.25\end{bmatrix},\ \text{Av}=\begin{bmatrix}0.37\\0.08\\0.33\\0.20\end{bmatrix},\ A^2\text{v}=\text{A(Av)}\begin{bmatrix}0.43\\0.12\\0.27\\0.16\end{bmatrix}\ \ldots\ldots.$$

$$A^6\text{v}=\begin{bmatrix}0.38\\0.13\\0.29\\0.19\end{bmatrix},\ A^7\text{v}=\begin{bmatrix}0.38\\0.12\\0.29\\0.19\end{bmatrix},\ A^8\text{v}=\begin{bmatrix}0.38\\0.12\\0.29\\0.19\end{bmatrix}$$

We notice that the sequence of iterates v, Av, A^2v,.... A^kv tends to the equilibrium value

$$\text{v}^* = \begin{bmatrix}0.38\\0.12\\0.29\\0.19\end{bmatrix}.$$ This is the PageRank vector of our web graph.

The Linear algebra approach

We can also look at this with the help of linear algebra. Let us denote by x_1, x_2, x_3, x_4 the importance of the four pages. Analyzing the situation at each node, we get the system:

$x_1= 1.\ x_3 + 1/2\ .\ x_4$
$x_2= 1/3\ .\ x_1$
$x_3=1/3\ .\ x_1 +1/2\ .\ x_2 + 1/2\ .\ x_4$
$x_4= 1/3\ .\ x_1 + 1/2\ .\ x_2$

This is equivalent to finding solutions of the equation $Ax=x$ where $A=\begin{bmatrix}0&0&1&1/2\\1/3&0&0&0\\1/3&1/2&0&1/2\\1/3&1/2&0&0\end{bmatrix}$

Calculating eigenvectors of the above equations, we get eigenvectors of the form c: $\begin{bmatrix}12\\4\\9\\6\end{bmatrix}$

Since PageRank should reflect only the relative importance of the nodes and since the eigenvectors are just scalar multiples of each other, we can choose any of them to be the PageRank vector. Choose v* to be the unique eigenvector with the sum of all entries equal to 1. This holds for c= 1/31 thus giving us our page rank vector. We can also understand the same by a probabilistic approach. Since the importance of a web page is measured by its popularity i.e. the number of links it has, we can view the importance of page i as the probability that a random surfer on the internet that opens a browser to any page and starts following the hyperlinks (Refer to the first figure), will visit the page i. The calculations remain the same as in the first approach, but the interpretation varies.

Say, a random surfer that is currently viewing webpage 2, has probability 1/2 to go to page 3 and 1/2 to go to page 4. We can model the process as a random walk on graphs. Each page has probability 1/4 to be chosen as the starting point. So the initial probability distribution is given by the column vector $\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$. The probability that page i will be visited after one step will be Ax and so on. The probability that page i will be visited after k steps is A^kx. The sequence Ax, A^2x... A^kx.. converges to a unique probabilistic vector v* called the stationary distribution and will be the PageRank vector. Moreover, the entry at the i position in the vector v* is simply the probability that at each moment at random surfer visits the page i. The PageRank vector we have calculated by different methods indicates that page 1 is the most relevant page. If we take a look at the graph, we see that node 3 has only one outgoing link to node 1 so it transfers all of its importance score to node 1.

The web is not one simple graph. It certainly has more than four pages. And it is extremely heterogeneous in nature. This gives rise to certain problems. The most common ones are Dangling nodes and Disconnected graphs.

Dangling Nodes and Disconnected graphs

Dangling nodes are the nodes which do not have any outgoing edges. Extending our simple example, suppose that there some pages that do not have any out-links (we call them dangling nodes), our random surfer will get stuck on these pages, and the importance received by these pages cannot be propagated. In the other scenario, if our web graph has two disconnected components, the random surfer that starts from one component has no way to get into the other component. All pages in other component will receive 0 importance. An example is shown in Figure 3. Similarly, we can have disconnected components where one graph is not connected to another page of another graph and hence leaves the surfer stranded on one graph unable to surf further (See Figure 4).

In order to deal with these two problems, we first need to know about the Random Surfer model. Say, A random surfer visits a web page with a certain probability which derives from the page's PageRank. The probability that the random surfer clicks on one link is solely given by the number of links on that page. This is why one page's PageRank is not completely passed on to a page it links to, but is divided by the number of links on the page.

So, the probability for the random surfer reaching one page is the sum of probabilities for the random surfer following links to this page. Now, this probability is reduced by the damping factor d. The justification within the Random Surfer Model, therefore, is that the surfer does not click on an infinite number of links, but gets bored sometimes and jumps to another page at random. The probability for the random surfer not stopping to click on

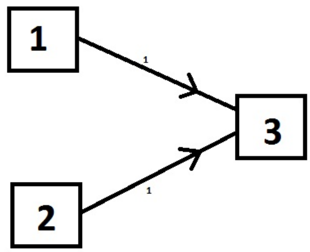

Figure 3: Dangling Nodes

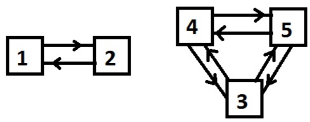

Figure 4: Disconnected graph

links is given by the damping factor d, which depends on the degree of probability therefore, set between 0 and 1. The higher d is, the more likely will the random surfer keep clicking links. Since the surfer jumps to another page at random after he stopped clicking links, the probability therefore is implemented as a constant $(1-d)$ into the algorithm. Regardless of inbound links, the probability for the random surfer jumping to a page is always $(1-d)$, so a page has always a minimum PageRank.

Based on the introduction damping factor d, Now we modify previous transition matrix as $M = (1-d)P + dR$, where $R = 1/N$. $\begin{bmatrix} 1 & 1 & . & . & . & 1 \\ . & . & & . & & . \\ . & . & & & . & . \\ . & . & & & & . \\ 1 & 1 & . & . & . & 1 \end{bmatrix}$

This new transition matrix models the random walk as follows: most of the time, a surfer will follow links from a page if that page has outgoing links. A smaller, but positive 3 percentage of the time, the surfer will dump the current page and choose arbitrarily a different page from the web, and "teleport" there. The damping factor d reflects the probability that the surfer quits the current page and teleports to a new one. Since every page can be teleported, each page has $1/n$ probability to be chosen. This justifies the structure of R. Mathematically, once we have M, computing the eigenvectors corresponding to the eigenvalue 1 is, at least in theory, a straightforward task. We would just need to solve the system $Ax = x$. But when the matrix M has size 30 *billion* (as it does for the real Web graph), it becomes an overwhelming task. An alternative way of computing the probabilistic eigenvector corresponding to the eigenvalue 1 is given by the Power Method.

Power Method Convergence

Theorem: Let M be a positive, column stochastic nxn matrix. Denote by v* its probabilistic eigenvector corresponding to the eigenvalue 1. Let z be the column vector with all entries equal to $1/n$. Then the sequence z, Mz, ..., M^kz converges to the vector v*.

The theorem guarantees that the method works for positive, column stochastic matrices. We reasoned that the iteration process corresponds to the way importance distributes over the net following the link structure. Computationally speaking, it is much more easier, starting from the vector with all entries 1, to multiply x, Mx ,..., M^nx until convergence then it is to compute the eigenvectors of M.

In fact, in this case, one needs to only compute the first couple of iterates in order to get a good approximation of the PageRank vector. For a random matrix, the power method is known to converge slowly, in general. What makes it work fast in this case however is the fact that the web graph is sparse. This means that a node i has a small number of outgoing links (a couple of hundred at best, which is extremely small corresponding to the 30 billion nodes it could theoretically link to). Hence the transition matrix A has a lot of entries equal to 0.

Conclusion

In a nutshell, PageRank is a very simple application of linear algebra. But when a simple calculation is applied hundreds of billions of times over the results, it can seem complicated. However, it has changed the way we look at surfing nowadays to the extent that "google" itself has become a widely used verb. There are many algorithms that search engines use. Google has most recently come up with a new algorithm called "The Humming Bird". It looks at PageRank along with other factors like whether Google believes a page to be of good quality, the words used on it and many other things.

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TORRICELLI'S TRUMPET: A PARADOX

ESHA SAHA

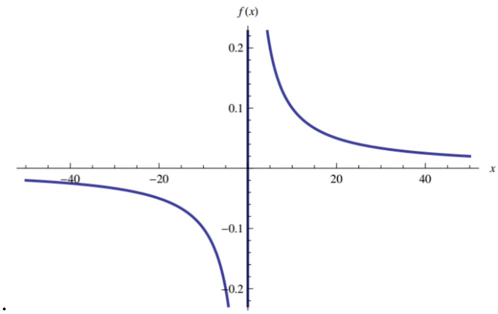
Abstract. The paradox is concerned with the volume and surface area of the Torricelli's Trumpet (also known as the Gabriel's Horn) - a hornlike object created by revolving the function $f(x) = \frac{1}{x} \forall x \in [1, \infty)$ about the x- axis. It has finite volume but infinite surface area. The paper is divided into three sections. Section 1 provides the definition of a paradox with explanation. Section 2 explains the nature of the paradox and gives the proof of the claim along with its converse. The question of other solids having similar properties as that of the Gabriel's Horn is explored in Section 3. In the end, based on all the findings, some conclusions are drawn.

Introduction

It is extremely important to understand what is meant by a paradox and how it is distinct from a fallacy, contradiction and an undecidable problem before moving on to the Gabriel's Horn Paradox. A paradox is a statement that apparently contradicts itself and yet might be true. A fallacy however is a statement that appears to be true but is false. For example, all infinite sets have the same number of elements in it. In computational complexity theory and computability theory, an undecidable problem is a decision problem for which it is impossible to construct a single algorithm that leads to a correct yes or no answer. Paradoxes can be of various types such as veridical paradox, self-reference paradox, infinite regress etc. Paradoxes appear in all disciplines, be it literature, philosophy, physics or mathematics and may sometimes lead to evolving of theories and clarification of definitions to avoid further contradictions and confusions.

The Torricelli's Trumpet

Evangelista Torricelli (Fleron, 1999) was a student of Galileo and is best known for his contribution to Physics and the discovery of barometer. He himself was surprised at this infinitely long solid having a finite volume in spite of having an infinite surface area. It is also called Gabriel's Horn, Archangel Gabriel being the angel blowing the horn to announce Judgement Day associating the finite with infinite (divine). This paradox is also known as the painter's paradox as the infinite surface of the solid can be painted with a finite supply of paint. If the solid is filled with paint and since the volume is finite, the inner surface would get completely painted.



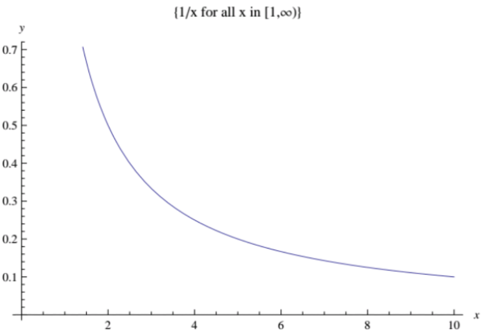


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Figure 1: $f(x) = \frac{1}{x}, x \in (-\infty, \infty)$ [1] Figure 2: $f(x) = \frac{1}{x} \forall x \in [1, \infty)$

It is important to understand the graph of the function $f(x) = \frac{1}{x} \forall x \in (-\infty, \infty)$ which is a hyperbola as in Figure 1. A hyperbola is the locus of all those points in the plane, the difference of whose distances from two fixed point(the foci) gives a positive constant (Weisstein, 2015a). The domain of the function has been restricted to $x \in [1, \infty)$ to avoid the asymptote at $x = 0$. When the domain of the function used to form the solid is $[1, \infty)$, the graph of the function is restricted to the first quadrant as in Figure 2.

Proof: Infinite surface with finite volume

Let $f : [1, \infty) \to \mathbb{R}$, $f(x) = 1/x$.
then the solid formed by revolving the graph of $f(x) = 1/x$ about the $x - axis$ is given in Figure 3. Let $a \in [1, \infty)$, then we calculate the volume and surface area between $x = 1$ and $x = a$, when a tends to infinity.

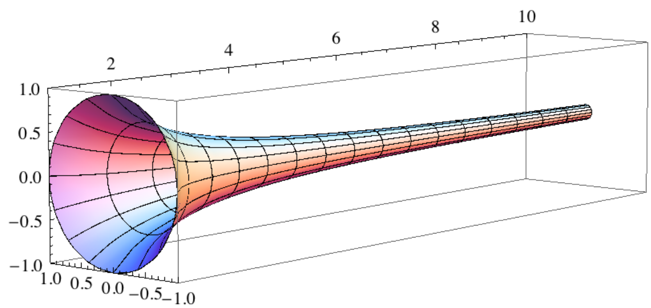

Figure 3: Solid of revolution

Part 1. Surface area

First we prove that the surface area of the solid approaches infinity. Surface area (henceforth denoted by SA) of the solid is given by

[1]All figures generated using Wolfram Mathematica 9

$$SA = \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx.$$

$$\text{Then SA}= \int_1^a 2\pi \tfrac{1}{x} \sqrt{1 + \left(\tfrac{-1}{x}\right)^2} dx > 2\pi \int_1^a \tfrac{1}{x} dx = 2\pi \log a$$

$$\text{Now SA}> 2\pi \log a \text{ and as } \lim_{a\to\infty} 2\pi \log a \to \infty$$

$$\Rightarrow SA \to \infty.$$

Hence, the surface area of the solid is infinite.

Part 2. **Volume**

Now we prove the statement of the paradox i.e. the volume of this solid is finite, given that the surface area is infinite. The volume V of the solid is given by

$$V = \int_1^a \pi f(x)^2 \, dx,$$

$$\text{i.e. } V = \int_1^a \pi \, \left(\tfrac{1}{x}\right)^2 = \pi \left(1 - \tfrac{1}{a}\right). \text{ (Anton, et al., 2002; Weisstein, 2015b)}$$

$$\text{Now } \lim_{a\to\infty} V = \pi$$

Hence, the volume of the given solid is finite.

Converse of the above property is false.

Proof: Now the converse of the above property is not true i.e. if the surface area is finite then the volume will be finite (Gabriel's Horn Wikipedia, 2015). This can be proved mathematically.

Let $f : [1, \infty) \to \mathbb{R}$ be a continuously differentiable function. Then the surface area of the solid of revolution of, $y = f(x)$ about the $x - axis$ is finite and so is the volume. The result is proven next.

Since the surface area SA is finite then:

$$\begin{aligned} S = \lim_{t\to\infty} \left(\sup_{x\geq t} f(x)^2 - f(1)^2\right) &= \lim_{t\to\infty} \sup \int_1^t f(x)^2 \, dx \\ &\leq \int_1^\infty \left| f(x)^2 \right|' dx = \int_1^\infty 2f(x) \left| f'(x) \right| dx \\ &\leq \int_1^\infty 2f(x) \sqrt{1 + f'(x)^2} dx = \tfrac{SA}{\pi} < \infty \end{aligned}$$

Hence, $\exists$ y_0 such that supremum $\sup \{f(x) | x \geq y_0\}$ is finite. As a result, $M = \sup \{f(x) | x \geq 1\}$ must be finite. Moreover, f is a continuous function on $[1, t_0]$, f is bounded on $[1, t_0]$. Therefore, volume $V = \int_1^\infty f(x) \pi f(x) \, dx \leq \int_1^\infty \frac{M}{2} 2\pi f(x) \, dx \leq \frac{M}{2} \int_a^b 2\pi f(x) \sqrt{1 + f'(x)^2} dx = \frac{M}{2}.SA$, and hence the volume is finite when the surface area is finite.

Gabriel's Wedding Cake

This section deals with another interesting solid called Gabriel's wedding cake that follows the same property as that of the Gabriel's Horn. The figure is generated by revolving the graph of the function $g(x) = \frac{1}{n} for, n \leq x < n + 1$ for $n = 1, 2, 3, ...$ about the $x - axis$ as given in Figure 4. The solid so formed appears to be a cake of infinitely many layers and hence the name.

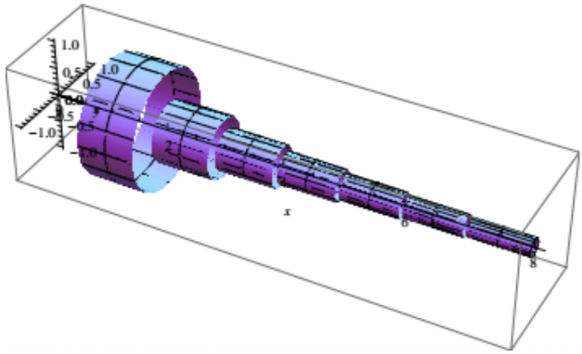

Figure 4: Gabriel's Wedding Cake

Proof: The surface area of the $n-th$ top is $\pi\left(\frac{1}{n}\right)^2-\pi\left(\frac{1}{n+1}\right)^2$. Therefore the total area of the annular tops is given by

$$A=\sum_{n=1}^{\infty}\left(\pi\left(\frac{1}{n}\right)^2-\pi\left(\frac{1}{n+1}\right)^2\right)=\pi=\pi\left(1\right)^2$$

Hence the resulting top layer will be a complete disk of radius 1. The total surface area is given by:

$$SA=\sum_{n=1}^{\infty}2\pi\left(\frac{1}{n}\right)(1)=2\pi\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$$

This series is divergent as the $p-series$ with $p=1$, $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ is divergent. Hence the surface area is infinite.

Now as each layer is simply a cylinder, the volume of each of the layers is πr^2, r being the radius of each cylinder. Hence, the volume of the $n-th$ cylinder will be $\pi\left(\frac{1}{n^2}\right)$. The total volume$(V)$ of the cake will be:

$$V=\pi\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

This is the $p-series$ with $p=2$ and converges by integral test for convergence for series.

$$\text{Consider }\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)=1+\frac{1}{2^2}+\frac{1}{3^2}+\dots$$

Euler found out this sum to be equal to be $\frac{\pi^2}{6}$ and hence the volume $V=\frac{\pi^3}{6}$, which is finite (Fleron 1999). Therefore, it can be concluded that Gabriel's wedding cake is another solid having infinite surface area and finite volume.

Conclusions

This paper has discussed the paradox of Torricelli's trumpet. It has been proved mathematically that it is possible to have solids that have infinite surface area but finite volume. The converse of this property is not true. A quick test on the function $f(x)=1/x^2$, which has a similar trend as that of the function in Figure 2 does not satisfy the conditions of the paradox presented in this paper.

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THE EVOLUTIONARY GAMES

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Abstract. Game Theory is the study of strategic decision making and has been used time and again . It was first explicitly applied to biology in the 1970s, although similar developments go back at least as far as the 1930s. This paper will focus on the application of game theory to evolving populations of life forms in biology. We will in particular study the Hawk-Dove Model, wherein we will make certain assumptions to bring the model closest to reality. This model, after making suitable amends, can be extended to the human society which would then analyze the behavioural patterns of a bully and a non-agressive person when confronted.

Introduction

Evolutionary game theory was first developed by R. A. Fisherin in his attempt to explain the approximate equality of the sex ratio in mammals. Evolutionary game theory originated as an application of the mathematical theory of games to biological contexts, arising from the realization that frequency dependent fitness introduces a strategic aspect to evolution.

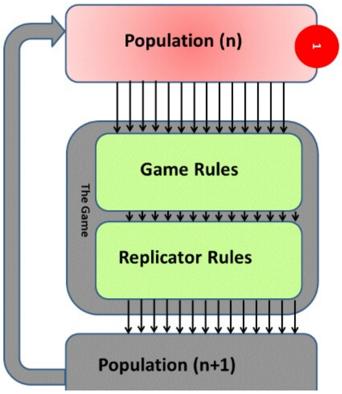
The Evolutionary Game Mode

It is important to realize that EGT is not just a specialist mathematical treatment of animal contests to determine their dynamics and results, but in a manner similar to the field of evolutionary algorithm. Evolutionary game theory transposes Darwinian mechanisms into a mathematical form by adopting a System Model of evolutionary processes with three main components - Population, Game, and Replicator Dynamics. The system process itself has four phases

- The first stage involves a population P(n) competing against each other. In this model the competition is represented by the game. The population will exhibit variation among competing individuals.
- In the next phase the 'Game' tests the strategies of the individuals under the "rules of the game". These rules produce different payoffs - in units of Fitness i.e. the production rate of offspring. Individuals in the population adopt different types of strategies to be the 'fittest' and hence these mixed strategies directly affect the payoff results.
- Based on this resulting fitness each member of the population then undergoes replication or culling determined by the exact mathematics of the Replicator Dynamics Process. This overall process then produces a New Generation P(n+1). Each surviving individual now has a new fitness level determined by the game result.
- The process is then repeated with the new generation taking the place of the previous one. It is an iterative process.

Evolutionary Stability

The main aim is to determine when and under what conditions, the evolutionary stability is achieved by the population while playing the games. The whole process can be summarised as follows :



Game Model

- A large population of individuals are randomly matched in pairs to play a symmetric and finite two-player game, where initially, all individuals always play the same (pure or mixed) strategy.
- Suddenly, a small population share switch to another strategy.
- If the residents on average do better than the mutants, then the resident strategy is evolutionarily stable against that mutation.

A strategy is evolutionarily stable if it is evolutionarily stable against all mutations.

Different Approaches to Evolutionary Game Theory (EGT)

The first approach derives from the work of Maynard Smith and Price and employs the concept of an evolutionarily stable strategy as the principal tool of analysis. It can thus be thought of as providing a static conceptual analysis of evolutionary stability.

The second approach constructs an explicit model of the process by which the frequency of strategies change in the population and studies properties of the evolutionary dynamics within that model.

As an example of the first approach, consider the problem of the Hawk-Dove game, analysed by Maynard Smith and Price in "The Logic of Animal Conflict".

In this game, two individuals compete for a resource of a fixed value V>0. (In biological contexts, the value V of the resource corresponds to an increase in the Darwinian fitness of the individual who obtains the resource; in a cultural context, the value V of the resource would need to be given an alternate interpretation more appropriate to the specific model in hand.) Each individual follows exactly one of two strategies described below:

Hawk	Initiate aggressive behaviour, not stopping until injured or until one's opponent backs down.
Dove	Retreat immediately if one's opponent initiates aggressive behaviour.

If we assume that,

- Whenever two individuals both initiate aggressive behaviour, it eventually results in conflict and the two individuals are equally likely to be injured.

- The cost of the conflict reduces individual fitness by some constant value C>0
- When a Hawk meets a Dove, the Dove immediately retreats and the Hawk obtains the resource and when two Doves meet the resource is shared equally between them.

Then the fitness payoffs for the Hawk-Dove game can be summarized according to the following matrix:

	Hawk	Dove
Hawk	$\frac{1}{2}$(V-C)	V
Dove	0	$\frac{V}{2}$

The payoffs listed in the matrix are of a player using the strategy in the appropriate row, playing against someone using the strategy in the appropriate column. (For example, if you play the strategy Hawk against an opponent who plays the strategy Dove, your payoff is V; else it is 0.)

Let Δ F denote the change in fitness for an individual following strategy s_1 against an opponent following strategy s_2. Also let F(s) denote the total fitness of an individual following strategy s.

Suppose that each individual in the population has an initial fitness of F_0. If σ is an evolutionarily stable strategy and μ is a mutant attempting to invade the population, then

$$\mathrm{F}(\sigma) = F_0 + (1p)\ .\ \Delta\ \mathrm{F}\ (\sigma,\sigma) + p\ .\ \Delta\ \mathrm{F}(\sigma,\mu)\ \text{........(I)}$$
$$\mathrm{F}(\mu) = F_0 + (1p)\ .\ \Delta\ \mathrm{F}(\mu,\sigma) + p\ .\ \Delta\ \mathrm{F}(\mu,\mu)\ \text{..........(II)}$$

where p is the proportion of the population following the mutant strategy μ.

Since σ is evolutionarily stable, the fitness of an individual following σ must be greater than the fitness of an individual following μ, otherwise the mutant following μ would be able to invade, and so

$$\mathrm{F}(\sigma) > \mathrm{F}(\mu)$$
$$\text{Now as } p \to 0$$
$$\Rightarrow \Delta\ \mathrm{F}(\sigma,\sigma) > \Delta\ \mathrm{F}(\mu,\sigma) \text{ or, } \Rightarrow \Delta\ \mathrm{F}(\sigma,\sigma) = \Delta\ \mathrm{F}(\mu,\sigma) \text{ And } \Delta\ \mathrm{F}(\sigma,\mu) > \Delta\ \mathrm{F}(\mu,\mu)$$

In other words, it means that a strategy σ is an ESS if one of two conditions holds:

- σ does better playing against σ than any mutant does playing against σ or,
- some mutant does good playing against σ just as well as σ does, but σ does better playing against the mutant than the mutant does.

Hence we can conclude that:

- If V>C , then the strategy Hawk is evolutionarily stable.
- If V<C , then there is no evolutionarily stable strategy if individuals are restricted to following pure strategies, although there is an evolutionarily stable strategy if players may use mixed strategies.

A mixed strategy would involve that the fitness of both the players i.e. Hawk and Dove is same. Consider the mutant strategy to be Hawk. Then, From (I) and (II) we get

$$\mathrm{W(Hawk)} = (1/2)(\mathrm{V\text{-}C}) + (1\text{-}\mathrm{p})(\mathrm{V})$$
$$\mathrm{W(Dove)} = \mathrm{p}(0) + (\mathrm{V}/2)$$

where 'W' describes the fitness and 'p' is the polpulation of the mutants i.e. Hawks

Equating the above two equations we get that stability is achieved in a situation having a mix of the two strategies where the population of Hawks is V/C.

The figure below is the solution for V=2, C=10 and fitness starting base B=4. The fitness of a Hawk for different population mixes is plotted as a black line, that of Dove in red. An ESS (a stationary point) will exist when Hawk and Dove fitness are equal: Hawks are $\frac{1}{5}$ of population and Doves are $\frac{4}{5}$ of the population.

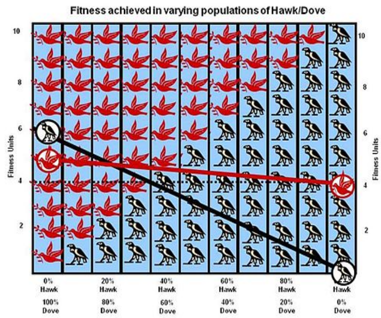


Figure 2

Conclusion

Clearly, Game theory is vast and its applications are numerous. Evolutionary Game theory itself spreads to many models of evolution. It is a major vehicle to help understand and explain some of the most fundamental questions in biology including the issue of group selection, altruism, parental care, co-evolution, and ecological dynamics.

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