ÉCLAT

MATHEMATICS JOURNAL

LADY SHRI RAM COLLEGE FOR WOMEN

VOLUME -5

2013-2014

PREFACE

Éclat, with its roots in French, means brilliance.The journey of *Éclat* gave us an enriching experience and sometimes even tested our brilliance. We have immense pleasure in bringing out its 5th volume. This journal aims at providing a platform for students who wish to publish their ideas and also other concepts they might have come across. To present diverse concepts, the journal has been divided into four sections - History of Mathematics, Rigour in Mathematics, Extension of Course Contents and Interdisciplinary Aspects of Mathematics. The work contained here is not original but consists of the review and expository articles contributed by both faculty and students. We expect that this volume will enhance the knowledge of students and stimulate them to get into the various realms of Mathematics.

The entire department of mathematics of our college has been instrumental in the publication of this journal. Its compilation has evolved after continuous research, discussion and rigorous efforts. We hope that this journal would keep on encouraging the students to hone their skills in doing individual research and in writing academic papers. It is an opportunity to go beyond the prescribed limits of the text and to expand the knowledge of the subject. We would like to thank all the authors who have contributed their papers/ideas in this volume of *Éclat*.

Also, we express our sincere thanks to the faculty advisors without whose guidance the efforts made couldn't have been possible to come to in its present form.

Contents

Topics	Page
1) History of Mathematics	1
• Grigori Perelman: The Genius in Hiding Shambhavi Gupta	3
• Sir William Rowan Hamilton: A Man Of Shades Vidushi Singh and Palak Gupta	9
2) Rigour in Mathematics	15
• A Multi-Echleon Multi Product Profit Oriented Closed Loop Network Design Jyoti Darbari, Vernika Agarwal and PC Jha	17
• Robe's Restricted Problem Of 2+2 Bodies Revisited Bhavneet Kaur	27
3) Extension of Course Contents	35
• Different Paths Leading To One Target Monika Singh	37
• Euclidean Geometry and The Parallel Postulate Kritibha Rai	45
• Knot Thoery: Knot Invariants Urvashi Negi and Garima Yadav	51
4) Inter-Disciplinary Aspects of Mathematics	55
• Game Theory and Business Strategies Nupur Sood	57
• Linear Time Invariant Theory Aditi Jain and Mansi Verma	63
• Mathematics in Solving Crimes Ankita Tulshyan	67

History of Mathematics

Mathematics is the oldest academic discipline involving stimulating and intriguing concepts. It is far beyond the ken of one individual, and to make any contribution to the evolution of ideas, an understanding of the motivation behind the ideas is needed. The section covers the genesis of mathematical ideas, the stream of thought that created the problem and what led to its solution. The aim is to acquaint the readers with historically important mathematical vignettes and make them inured in some important ideas of Mathematics.

GRIGORI PERELMAN: THE GENIUS IN HIDING

SHAMBHAVI GUPTA

Abstract. There have been many great mathematicians, who have made path breaking discoveries in mathematics. But the most unusual amongst them is Grigori Perelman. Perelman made a breakthrough in the field of Mathematics by his proof of the Poincarè conjecture. Meet the genius who declined a million dollars and the Fields medal. Despite his withdrawl from public life, he has been the subject of much media speculation. All seek to answer the burning question: What would drive a man to shun his well-deserved fame and recognition? Let us unravel the mystery man, the genius in hiding.

Introduction

Grigori Yakovlevich Perelman was born in Leningrad, Soviet Union (now St. Petersburg, Russia) on 13 June 1966, to Jewish parents. His father, Yakov, was an electrical engineer and his mother, Lyubov, was a mathematics teacher. His father had a major influence in developing his problem solving skills. Speaking about his father, Perelman said:-
"He gave me logical and other math problems to think about. He got a lot of books for me to read. He taught me how to play chess. He was proud of me"

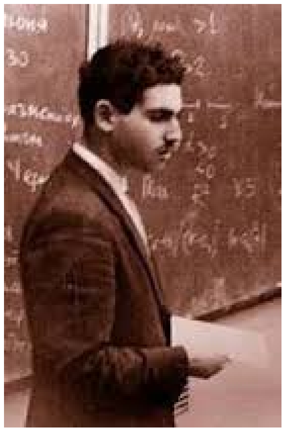
Student Life, 1980's

Early Life and Education

Perelman's mother, Lyubov gave up graduate work in mathematics to raise him and also helped develop his mathematical skills.

Perelman's mathematical talent became apparent at the age of ten. He took part in district mathematics competitions and showed a marked talent. Perelman started to attend a mathematics club run by a nineteen-year-old coach named Sergei Rukshin, an undergraduate student at Leningrad University.

Rukshin quickly saw Grigori's potential and hence a strong bond developed between them. In the summer of 1980, Rukshin tutored Perelman in English so that he could enter *Leningrad's Special Mathematics and Physics School Number* 239 (a specialized school with advanced mathematics and physics programs). Perelman excelled in all subjects except physical education. The highly talented mathematicians tutored by Rushkin were all put in the same class. At the school, their mathematics teacher was Valery Ryzhik. Apart from mathematics, Ryzhik ran a chess club, which Grigori attended. He showed considerable talent in the game. When he was fifteen, Perelman attended a summer camp run by Rukshin, which helped him develop his problem solving abilities in mathematics.



Perelman in Budapest,1982

In January 1982, he was selected to be a part of the team set to represent Soviet Union at *The International Mathematical Olympiad*. He attended a session in Chernogolovka, where the students were subjected to physical and mathematical training. Perelman not only achieved full marks at the session in Odessa, but also went on to achieve full marks at the International Mathematical Olympiad competition in Budapest. He received a gold medal and a special prize for achieving a perfect score. Being a member of the Soviet team gave Perelman direct entry to university.

Perelman entered *Leningrad State University* in autumn 1982. At that time, *the Leningrad Department of the Steklov Mathematics Institute of USSR Academy of Sciences* was under Ivan Vinogradov's leadership and accepted no Jews. Aleksander Danilovic Aleksandrov wrote to the director requesting that Perelman be allowed to undertake graduate work under his supervision and hence graduation was granted. Yuri Burago was his official advisor. His university work was exceptional and he graduated in 1987. He had already published a number of papers:

- Realization of abstract k-skeletons as k-skeletons of intersections of convex polyhedra in R^{2k-1} (Russian) (1985); (with I V Polikanova).
- A remark on Helly's theorem (Russian) (1986)

- A supplement to A D Aleksandrov's, 'On the foundations of geometry'(Russian) (1987), in which Perelman discussed the equivalence of a Pasch-style axiom of Aleksandrov and some of its consequences.
- On the k-radii of a convex body (Russian) (1987).

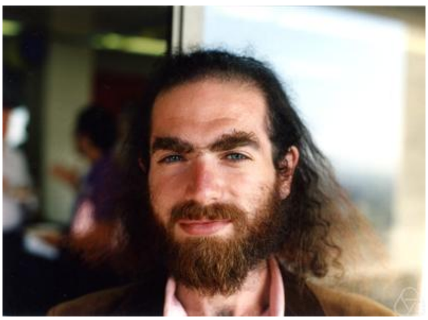
Perelman went on to earn a Candidate of Sciences degree (the Soviet equivalent of Ph.D.) at the *School of Mathematics and Mechanics of the Leningrad State University*, one of the leading universities in the former Soviet Union. His thesis was titled *Saddle surfaces in Euclidean spaces*(1990). He had already published one of the main results of the thesis in an example of a complete saddle surface in R^4 with Gaussian curvature bounded away from zero (1989).

In 1991 Perelman won the "Young Mathematician Prize of the St. Petersburg Mathematical Society" for his work on Aleksandrov's spaces of curvature bounded from below.

Burago contacted Mikhael Leonidovich Gromov, who had been a professor at *Leningrad State University* but was at this time a permanent member of the *Institut des Hautes tudes Scientifiques* outside Paris, for recommending Perelman. The said invitation allowed Perelman to spend several months at IHES working with Gromov on Aleksandrov spaces.

After visiting the IHES he returned to the *Steklov Mathematics Institute* but through the efforts of Gromov, Perelman was invited to the United States to talk at the 1991 Geometry Festival held at Duke University, North Carolina. He lectured on Aleksandrov spaces with Burago and Gromov. Perelman's first major paper, written jointly with Burago and Gromov, was *'A D Aleksandrov spaces with curvatures bounded below'* (1992).

Perelman was invited in 1992 to spend the autumn semester at the Courant Institute, New York University, and for the spring 1993 semester at Stony Brook, State University of New York, both funded by a fellowship.
Thereafter, he accepted a two-year Miller Research Fellowship at the University of California, Berkeley in 1993. He published some remarkable papers during these years. *Elements of Morse theory on Aleksandrov spaces (1993)* investigates the local topological structure of Aleksandrov spaces. A manifold of positive Ricci curvature with almost maximal volume (1994) solves a conjecture about a complete Riemannian manifold M_n.



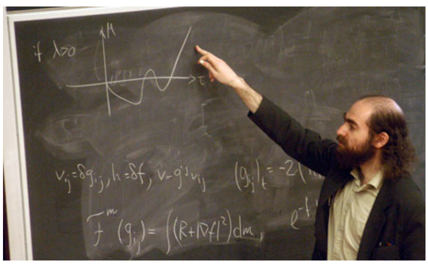
Grigori in UC, Berkeley, 1993

The biggest breakthrough, however, was his paper, *Proof of the soul conjecture of Cheeger and Gromoll (1994).* Perelman was invited to address the International Congress of Mathematicians in Zurich in 1994, where he gave a lecture on 'Alexandrov spaces with curvature bounded below'. After having proved the Soul conjecture in 1994, he was offered jobs at several top universities in the US, including Princeton and Stanford, but he rejected all of them and returned to the Steklov Institute in St. Petersburg in the summer of 1995, for a research-only position. In 1996, he refused to accept the *European Mathematical Society* prize.

His Contribution To Mathematics

Grigori Perelman made landmark contributions to *Riemannian geometry, Aleksandrov geometry and Geometric topology.* In 1994, Perelman proved the soul conjecture (conjectured by J Cheeger and D Gromoll in 1972). In 2003, he proved Thurston's geometrization conjecture (conjectured in 1970). This consequently solved the Poincarè conjecture; posed in 1904 by Henri Poincarè, before which its solution was viewed as one of the most important and difficult open problems in topology.

Grigori presented the proof of Poincarè conjecture in three papers made available in 2002 and 2003. The inspiration for the proof came from his discussions with Richard Hamilton, whose lecture he attended at the institute for advanced study. Hamilton used Ricci flow to attack the problem.



Perelman explaining Poincarè conjecture, 2003

The three preprints posted on the arXiv in 2002-2003 are:-

(1) The entropy formula for the Ricci flow and its geometric applications;
(2) Ricci flow with surgery on three-manifolds; and
(3) Finite extinction time for the solutions to the Ricci flow on certain three-manifold, which proved the above conjectures.

Perelman's proof was rated one of the top cited articles in Math-Physics in 2008.

Grigori was awarded the Fields Medal (2006)-*For his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow.* Perelman refused the invitation to be a plenary speaker at the 2006 International Congress of Mathematicians (the first person to have done so). He was also awarded the Clay Millennium

Prize (2010) for the resolution of the Poincarè conjecture, which he rejected (although the ceremony took place at the Institut Ocanographique, Paris in his honor).

Notable Awards

Some of the notable awards conferred upon him are:

- Saint Petersburg Mathematical Society Prize (1991) (accepted).
- European Mathematical Society Prize (1996), (declined).
- Fields Medal (2006), (declined); Millennium Prize (2010), (declined).

Withdrawal from Mathematics

Perelman's Character and Persona

(1) **No greed for name, fame or money**: Perelman told Sir John Ball, who was then the president of the International Mathematical Union, that his reason for refusing the prize was that he was not interested in fame or money. He did not consider himself a hero of mathematics. His reasons for refusing the medal, according to Ball, are complex, but *"it centers around his feeling of isolation from the mathematical community, and not wanting to be seen as a figurehead."* All that mattered to him was that the proof was correct. In 2010, when he was considered for prize by Clay Institute, he turned down the prize saying that he had all he wanted.

(2) **He is media phobic**: Perelman has avoided journalists and other members of the media. Masha Gessen, the author of *'Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century'*, a book about him, was unable to meet him.

(3) **Moral character and reasons for withdrawl from Mathematics**: In 2005, Perelman resigned as a senior researcher at the Steklov Institute. Perelman is quoted in an article in The New Yorker saying that he was disappointed by the ethical standards of the mathematical community. It is reported that he retired from mathematics, finding it too painful and political. In 2010, when he was considered for prize by Clay Institute, he told it would be unfair to not share the prize with Richard Hamilton, and stated that *"The main reason is my disagreement with the organized mathematical community. I don't like their decisions, I consider them unjust"*.

Conclusion

Let us end this article by quoting Mikhael Gromov:-

"Perelman has moral principles to which he holds. And this surprises people. They often say he acts strangely because he acts honestly, in a nonconformist manner, which is unpopular in this community - even though it should be the norm. "

It would not be wrong to say that Grigori Perelman wanted to be a servant of mathematics, with no greed for recognition.

References

[1] *M Gessen*, Perfect Rigor: A Genius and the Mathematical Breakthrough of the Century (New York, 2009).

[2] 2006 Fields Medals awarded, *Notices* Amer. Math. Soc. 53 (9) (2006)

[3] *J Lott*, The work of Grigory Perelman, International Congress of Mathematicians I (Eur. Math. Soc., Zrich, 2007)

[4] *D Mackenzie*, Breakthrough of the year. The Poincarè Conjecture-Proved, Science 314 (5807) (2006), 1848-1849. 5. Perelman, G. (1994)."Proof of the soul conjecture of Cheeger and Gromoll". Journal of Differential Geometry

SHAMBHAVI GUPTA, B.Sc.(H) MATHEMATICS, 4TH SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN

E-mail address: shambhavigupta.lsr@gmail.com

SIR WILLIAM ROWAN HAMILTON - A MAN OF SHADES

PALAK GUPTA AND VIDUSHI SINGH

Abstract. Sir William Rowan Hamilton was an Irish mathematician, astronomer and physicist, who made important contributions to classical mechanics, optics and algebra. His greatest contribution is perhaps the reformulation of Newtonian mechanics, now called *Hamiltonian mechanics*. In mathematics, he is perhaps best known as the inventor of *quaternions*. He had a versatile personality and fancied himself as a poet. He was titled as *the Lagrange of Ireland*. He lived a life of many fluctuations yet emerged to be a remarkable personality.

Introduction

At the stroke of midnight, between 3 and 4 August, 1805 was born the man who went on to become Ireland's greatest man of science: Sir Rowan Hamilton. Possibly because of his father's financial circumstances, Hamilton was sent to the village of Trim (40 miles north-west of Dublin) to live with his uncle , at the tender age of 3. His uncle, Reverand James Hamilton was an accomplished linguist and polymat. Hamilton was educated by his uncle, with whom he lived until he entered college.

Sir Rowan Hamilton

Early Life And Education

William began his education as soon as he arrived in Trim, quickly revealing himself to be a child prodigy. His uncle soon discovered that Hamilton had a remarkable ability to learn languages. By the age of ten, he had had learnt both classical and modern European languages, as well as Persian, Arabic, Hebrew, Hindustani, Sanskrit, and even Marathi and Malay.
At the age of 12, Hamilton met young calculating prodigy, Zerah Colburn, who could perform amazing mental arithmetical feats. Hamilton joined in competitions of arithmetical ability with him and one can say that losing to Colburn sparked Hamilton's interest in mathematics.
Hamilton's formal introduction to mathematics came at the age of 13 when he studied Clairaut's Algebra, a task made somewhat easier as Hamilton was fluent in French by this time.
At 15 years of age, he started studying the works of Newton and Laplace. In 1824, Hamilton found an error in Laplace's *Mcanique cleste* and, as a result of this, he came to the attention of John Brinkley, the Royal Astronomer of Ireland, who said:-
"This young man, I do not say will be, but is, the first mathematician of his age."
Hamilton entered Trinity College, Dublin at the age of 18. There he studied both classics and mathematics. He was first in every subject and at every examination amongst a number of extraordinary competitors. During his four years in Trinity, he demolished the competition in one examination after another. The college awarded him two separate 'optimes' in Classics, both for Greek and physics, for his performances: no optimes had been awarded for twenty years prior to this.
In 1827, in spite of his young age and lack of experience, he was appointed Andrews Professor of Astronomy, and consequently became *ex officio* Astronomer Royal of Ireland and director of the observatory at Dunsink, where he spent the rest of his life.

Career And Contributions

Hamilton made important contributions to the understanding of dynamics, classical mechanics and optics, invented quaternions and in graph theory, developed what he called the 'Icosian calculus'.

- His greatest contribution is perhaps the reformulation of Newtonian mechanics, now called *Hamiltonian mechanics.* This work has proven central to the modern study of classical field theories such as electromagnetism, and to the development of quantum mechanics.

- His first discovery, as a student, was in an early paper that he communicated in 1823 to Dr. Brinkley, who presented it under the title of 'Caustics', whose revised version was accepted under the title, *The Theory of System of Rays* and the first

part was printed in 1828 in the Transactions of the Royal Irish Academy. The second and third parts appeared in the three voluminous supplements (to the first part) which were published in the same Transactions, and in the two papers *On a General Method in Dynamics*, which appeared in the Philosophical Transactions in 1834 and 1835.
In these papers, Hamilton developed his great principle of Varying Action. The most remarkable result of this work is the prediction that a single ray of light entering a biaxial crystal at a certain angle would emerge as a hollow cone of rays. This discovery is still known by its original name, *'conical refraction'*.

- In 1827, Hamilton presented a theory of a single function, now known as *Hamilton's principal function*, that brings together mechanics, optics, and mathematics, which helped to establish the wave theory of light. He proposed for it when he first predicted its existence in the third supplement to his "Systems of Rays", read in 1832.

- The most important of Hamiltons major mathematical works is *The General Method in Dynamics.*

- Hamilton's "Theory of Algebraic Couples", published in 1835 by the Royal Irish Academy, contained his first ideas about algebra. He tried to develop an algebra for dealing with triples of the form :
$$a + ib + jc$$
where a, b, c are real and i, j are imaginary. On 16 October 1843 (a Monday), Hamilton was walking along the Royal Canal with his wife to preside at a Council meeting of the Royal Irish Academy. Although his wife talked to him now and again, Hamilton hardly heard her, for the discovery of the quaternions, the first non-commutative algebra to be studied, was taking shape in his mind:-
" And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples ... An electric circuit seemed to close, and a spark flashed forth."
He could not resist the impulse to carve the formulae for the quaternions
$$i^2 = j^2 = k^2 = ijk = -1$$
in the stone of Broome Bridge (or Brougham Bridge as he called it). The multiplication is associative but not commutative. In 1958 the Royal Irish Academy erected a plaque commemorating this. Hamilton felt this discovery would revolutionize mathematical physics and he spent the rest of his life working on quaternions. He wrote:-
"I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions [the calculus] was for the close of the seventeenth."

- Hamilton introduced, as a method of analysis, both quaternions and biquaternions, the extension to eight dimensions by introduction of complex number coefficients.

His work was assembled in 1853 in the book *Lectures on Quaternions*. Today, the quaternions are used in computer graphics, control theory, signal processing and in pure mathematics, with applications throughout algebra and geometry. His treatise, *Elements of Quaternions* was published posthumously in 1866. His ideas played a major role in the development of matrix algebra.

- Hamilton developed the variational principle, which was reformulated later by Carl Gustav Jacob Jacobi. He also introduced the Icosian game or Hamilton's puzzle which can be solved using the concept of a Hamiltonian path.

Commemorative stamp issued in 1943

Achievements And Acknowledgements

- He achieved the rare distinction of obtaining an *optime*, both for Greek and for physics in his college life. Hamilton might have attained many more such honours (he was expected to win both the gold medals at the degree examination), if he had not been appointed to the Andrews Professorship of Astronomy in the University of Dublin, a post vacated by Dr. Brinkley in 1827.
- He was awarded with, Royal Society Royal Medal and Fellow of the Royal Society of Edinburgh in 1835.
- Hamilton's Principle was acknowledged by the Royal Society of London, by Jacobi in particular where he referred Hamilton as *the illustrious Astronomer Royal of Dublin* and later as *the Lagrange of your country.*
- In 1837, he was elected as the *President of the Royal Irish Academy.*
- In 1843, he was awarded Civil List life Pension by the British Government.

Character And Personal Life

In his prime, Hamilton was a convivial and jovial man whose wide circle of friends included Agustus DeMorgan and John Herschel. Hamilton was ever courteous and kind in answering applications for assistance in the study of his works.
He was excessively precise and hard to please with reference to the final polish of his own works for publication; and it was probably for this reason that he published so little compared with the extent of his investigations. It was one of the peculiar characteristics of Hamilton's mind never to be satisfied with a general understanding of a question; he pursued the problem until he knew it in all its details.
During his lectures, he began with very elementary and obvious explanations and suddenly he would switch to material totally incomprehensible to the students.
When Hamilton toured England and Scotland, he met the poet Wordsworth and they became friends as Hamilton fancied himself as a poet. The two men had long debates over 'science versus poetry'. Hamilton liked to compare the two, suggesting that mathematical language was as artistic as poetry.
Hamilton had various romantic attachments but one in particular stands out. This was to Catherine Disney, with whom he became infatuated, but due to family pressure, when she married another man, he was disheartened. In desperation, he proposed to Helen Maria Bayly, who was pathologically shy and timid. They had three children, two sons and a daughter: William Edwin, Archibald Henry and Helen Eliza Amelia.

Death And Commemorations

Helen was unable to give him the domestic support he needed, as she suffered a serious illness that left her semi-invalid for the rest of her life. As time passed, the regime of irregular meals, chronic overwork and marital unhappiness began to take its toll, and the poor man increasingly sought solace from the bottle.
He was delighted to learn that he has been elected as the first foreign member of National Academy of Sciences of the United States of America.
He died on September 2, 1865 in his sixtieth year.

COMMEMORATION OF HAMILTON:

- Hamilton's equations are a formulation of classical mechanics.
- A commemorative coin was issued by the Central Bank of Ireland in his honour.
- Numerous other concepts and objects in mechanics, such as Hamilton's principle, Hamilton's principal function, and the Hamilton-Jacobi equation, are named after him.
- 'Hamiltonian' is the name of both a function (classical) and an operator (quantum) in physics, and is, in a different sense, a term from graph theory.
- Hamilton College (New York), a liberal arts college in Clinton, New York, is named in his honor.

Coin issued by Central Bank of Ireland

- The RCSI Hamilton Society was founded in his name in 2004.
- The algebra of quaternions is usually denoted by H, or in blackboard bold in honour of Hamilton.

Closing Note

Hamilton is recognized as one of Ireland's leading scientists and is increasingly celebrated. The Hamilton Institute is an applied mathematics research institute at NUI Maynooth and the Royal Irish Academy holds an annual public Hamilton lecture at which Murray Gell-Mann, Frank Wilczek, Andrew Wiles, and Timothy Gowers have all spoken.
The year 2005 was the 200th anniversary of Hamilton's birth and the Irish government designated it the 'Hamilton Year', to celebrate Irish science. Trinity College, Dublin marked the year by launching the *Hamilton Mathematics Institute.*
The application of quaternions in computer graphics, control theory, signal processing and orbital mechanics clearly depicts Hamiltons contribution to the advancement of technology. His name has definitely been written with golden words in the history of mathematics and astronomy.

References

[1] Makers of Mathematics - Stuart Hollingdale
[2] Remarkable Mathematicians Ioan James
[3] http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Hamilton.html

Palak Gupta and Vidushi Singh, B.Sc.(H) Mathematics, 4th Semester, Lady Shri Ram College for Women
E-mail address: palak3007gupta@gmail.com,vidushis23@gmail.com

Rigour in Mathematics

This section introduces advance Mathematics to the readers aiming at high standards of proofs. It stimulates interest and lays the foundation for further studies in different branches.

A MULTI-ECHELON MULTI PRODUCT PROFIT ORIENTED CLOSED LOOP NETWORK DESIGN

JYOTI DARBARI, VERNIKA AGARWAL AND PC JHA

ABSTRACT. Growing environmental concerns are drawing greater attention towards Closed-Loop Supply Chains (CLSCs). Unlike the traditional supply chains, CLSCs focus on product returns. An efficient and effective CLSC network works on the principle of recapturing value by considering appropriate value added product recovery options. This article considers a general closed loop supply chain network design for recovery options of returned products. In the paper, we consider a closed supply chain network which includes raw material supplier, manufacturer, distribution centers, customer zones, collection/inspection, fabrication, hybrid dismantling-component fabrication and recycling centers. In the forward flow, the model considers production, distribution of the new products to the customer zones and for the reverse flow it considers material recovery by recycling and value added product recovery by considering refurbishing and repairing. The returned products are disassembled and the components are sorted based on their utility. The fabricated components are used as new parts for manufacturing new products and for refurbishing of the returned products which are further sold in the secondary markets. A mixed integer linear programming model is formulated with an objective of maximizing the total expected profit.

INTRODUCTION

A supply chain is a network of supplier, production, retailers, and transportation channels organized to acquire raw materials, convert them to finished products, and distribute final products in an efficient way to customers (Pishvaeeet al. 2011). Closed-loop supply chain management as defined by Guide and Van Wassenhove (2009) is:

The design, control, and operation of a system to maximize value creation over the entire life cycle of a product with dynamic recovery of value from different types and volumes of returns over time.

A closed-loop supply chain (CLSCs) comprises of two parts: forward and reverse logistics. The forward flow encompasses activities related to the flow of materials and information starting from the suppliers and ending with the final delivery of the finished product to the customer. For the reverse logistics, the flow of returned products is processed from the customers back to the manufacturer to recapture value of for proper disposal. The closed loop supply chain network considered in the paper includes:

- raw material supplier,
- manufacturer,

- distribution centers,
- customer zones,
- collection/inspection(CIC),
- fabrication center,
- hybrid dismantling-component fabrication centre,
- disposal centre,
- recycling center,
- secondary markets and
- spare markets.

In the forward flow, the model considers assembly of the components and the flow of the new products to the customer zones through the distribution centres and for the reverse flow it considers value added product recovery, component recovery and material recovery. The amount of returned products is assumed to be a predefined fraction of the demand of each customer zone. The distribution centres also act collection centres and independent CIC also operate in customer zones which have no distribution centre. The returned products are sorted at the CIC based on their condition and are sent to either the fabrication centre or the hybrid centre for dismantling. The number of products sent to the fabrication center also depends on the demand of the fabricated products in the secondary markets. At the hybrid facility the returned products are disassembled and their components are sorted based on their utility. The components are either fabricated at the component fabrication unit of the same facility or are sent for recycling or disposal. The fabricated components which are as good as new components are used as new parts for manufacturing new products. A fraction of the remaining components are sent to the fabrication centre for refurbishing of the returned products which are further sold in the secondary markets, and the remaining are sold at the spare markets. The objective is to maximize the total expected profit by optimally determining the flow of returned products at each facility in the network.

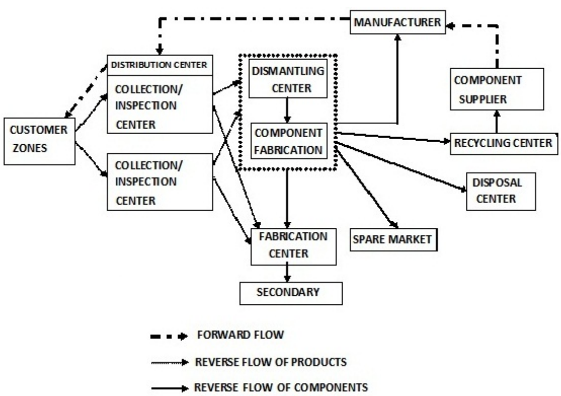


Figure1

Sets

We will use the following sets in our model:

c index of customers c=1, 2,....C;

l index of collection centers/inspection centers l=1, 2, ... L;

n index of products n=1,2, ...N

a index of components a=1,2, ...A

m index of secondary markets m=1, 2, ... M;

f index of spare markets f=1,2,....F;

j index of the distribution centers j=1,2....J.

Cost parameters

We will use the following cost parameters in our model:

CIC_n Per unit cost of inspection of n^{th} product at collection center/inspection center.

CDM_n Per unit cost of dismantling of n^{th} product at dismantling center.

CFC_n Per unit cost of fabrication of n^{th} product at fabrication center.

CCF_a Per unit cost of component fabrication of a^{th} component the component fabrication center.

CDC_a Per unit cost of disposal component of a^{th} component at disposal center.

CAC_n Per unit cost of assembling product n at manufacturing center center.

CPC_a Per unit cost of procurement of a^{th} component from component supplier.

Transportation costs:

TCL_{ncl} Unit transportation cost of the returned product n transported from customer zone c to collection center/inspection center l.

TFL_{nl} Unit transportation cost of the returned product n transported from collection center/inspection center l to fabrication center.

TL_{nl} Unit transportation cost of the recoverable product n transported from collection center/inspection center l to dismantling center.

TS Unit transportation cost of the disassembled components transported from disassembling center to disposal center.

TI Unit transportation cost of the fabricated components transported from component fabrication center to fabrication center.

TM Unit transportation cost of the fabricated components transported from component fabrication to manufacturing center.

TMD_{nj} Unit transportation cost of the product n transported from manufacturing center to distribution center j.

TDC_{njc} Unit transportation cost of product n transported from distribution center j to customer zone c.

Parameters

ρ_n Fraction of total units of nth product to be transported for fabrication from collection center l.
γ_a Fraction of total units of a^{th} component transported to component fabrication from disassemble center.
δ_a Fraction of total units of a^{th} component transported to recycling center from disassemble center.
ϕ_a Fraction of total units of a^{th} component transported to fabrication center from component fabrication center.
η_a Fraction of total units of a^{th} component transported to manufacturing center from component fabrication center.
λ_n Fraction of the total demand that is returned.
DEM_{mn} Demand of n^{th} product at m^{th} secondary market.
$PREV_{mn}$ Per unit revenue generated by n^{th} product at m^{th} secondary market.
$CREV_{af}$ Per unit revenue generated by a^{th} component at f^{th} spare market.
$RREV_a$ Per unit revenue generated by a^{th} component at recycling center.
$PPROF_n$ Selling price of n^{th} product at primary market.
$FDEM_a$ Demand of the a^{th} component at the fabrication center.
X_{cn} Units of n^{th} product returned to customer zone c.
C_{apl} Capacity of collection center /inspection center l.
Q_{na} $\begin{cases} 1, & \text{if } n^{th} \text{ product consist of } a^{th} \text{ component} \\ 0, & \text{otherwise} \end{cases}$
H_{jc} $\begin{cases} 1, & \text{if } j^{th} \text{ distribution center serves } c^{th} \text{ customer zone} \\ 0, & \text{otherwise} \end{cases}$

Decision Variables

RX_{nl} Units of n^{th} product collected at collection center l.
Z_n Units of n^{th} product transported from collection centers to dismantling center.
O_n Total number of returned units of n^{th} product transported to fabrication center.
B_{mn} Total number of returned units of n^{th} product transported from fabrication center to m^{th} secondary market.
V_a Total units of a^{th} component transported from disassemble center to recycling center.
G_a Total units of a^{th} component transported to component fabrication.
DIS_a Total units of a^{th} component transported to disposal center.
M_a Total units of a^{th} component transported from component fabrication center to manufacturing center.
RI_a Total units of a^{th} component transported from component fabrication center to fabrication center.
RF_{af} Total units of a^{th} component transported from component fabrication center to

f^{th} spare market.

CS_a Total units of a^{th} component provided by the supplier.

DC_{jn} The amount of units of n^{th} product at j^{th} distribution center.

D_n Total amount of units manufactured.

Y_{cl} $\begin{cases} 1, & \text{if customer zone c returns to collection center l} \\ 0, & \text{otherwise} \end{cases}$

P_{af} $\begin{cases} 1, & \text{if } a^{th} \text{ componet is sold to } f^{th} \text{ spare market} \\ 0, & \text{otherwise} \end{cases}$

$MDEM_a$ Demand of the a^{th} component at the manufacturing center.

Objective Function

The objective function (Maximization) is given by:

$$\text{Maximize Profit} = \text{TRG-TTC-TFC}$$

The objective function consists of the following :

- Total revenue generated:

$$TRG = \sum_m \sum_n PREV_{mn} B_{mn} + \sum_a \sum_f CREV_{af} RF_{af} + \sum_a PREV_a V_a + \sum_n D_n PPROF_n \tag{1}$$

The equation represents the revenue generated from (a) secondary markets;(b) spare markets; (c) the recycling of components and (d) primary market.

- Total transportation costs:

$$TTC = \sum_n \sum_c \{\sum_l TCL_{ncl}\} RX_{nl} + \sum_n \{\sum_l TFL_{nl}\} \sum_m B_{mn} + \sum_n \sum_l TL_{nl} Z_n +$$
$$\textstyle\sum_a \{TIRI_a + TSDIS_a + TMM_a\} + \sum_n \sum_j TMD_{nj} D_n + \sum_n \sum_j \sum_c TDC_{njc} DC_{nj} (2)$$

The equation represents the transportation cost incurred by (e) transporting returned products from customer zones to collection/inspection centers; (f) transporting products from collection/inspection centers to fabrication center; (g) transporting products from collection/inspection centers to dismantling center; (h) transporting fabricated components from component fabrication center to fabrication center; (i) transporting components from disassemble center to disposal; (j)

- Total facility/processing cost:

$$TFC = \sum_n (CAC_n D_n + CIC_n \sum_l RX_{nl} + CDM_n Z_n + CFC_n \sum_m B_{mn}) +$$
$$\textstyle\sum_a (CCF_a G_a + CPC_a CS_a + DIS_a CDC_a) (3)$$

The equation represents the cost parameters which include (k) cost of inspecting units; (l) cost of dismantling units; (m) cost of fabricating units; (n) cost of disposal and (o) cost of component fabricating.

Finally, the objective is to maximize profit which is the difference between the total revenue generated and the various costs namely transportation cost and cost incurred at various facilities.

Constraints

Following are the constraints :

- Collection/inspection Center

$$RX_{nl} = \lambda_n \sum_c X_{cn} Y_{cl} \quad \forall n, l \tag{4}$$

$$\sum_l Y_{cl} = 1 \quad \forall c \tag{5}$$

$$\sum_n RX_{nl} \leqslant Cap_l \quad \forall l \tag{6}$$

The constraint(4) determines the total units of each returned product collected at the collection center l irrespective of whether the collection center is operating single or it is integrated with distribution center. The constraint (5) ensures that each customer zone returns to only one collection center. The constraint (6) signifies the capacity restriction. The amount of units transported from customer zones to collection center should be less than or equal to the capacity of that collection center.

- Fabrication center and Secondary markets

$$O_n = \sum_l \rho_n RX_{nl} \quad \forall n \tag{7}$$

$$\sum_m B_{mn} \leqslant O_n \quad \forall n \tag{8}$$

$$B_{mn} \leqslant DEM_{mn} \quad \forall m, n \tag{9}$$

The constraints determine the total number of units of each product transported to fabrication center after initial inspection and after realizing the total demand at the secondary markets.

- Dismantling center

$$Z_n = \sum_l (1-\rho_n) RX_{nl} \quad \forall n \tag{10}$$

Here we determine the total number of units of each product transported from collection/inspection center to the dismantling center.

- Component fabrication center

$$G_a = \sum_n \gamma_a Q_{na} Z_n \quad \forall a \tag{11}$$

It determines the number of units that are transported to component fabrication center from dismantling center.

- Recycling center

$$V_a = \sum_n \delta_a Q_{na} Z_n \quad \forall a \tag{12}$$

This constraint determines amount of units of each component send to the recycling center from dismantling center

.

- Disposal center

$$DIS_a = \sum_n (1-\gamma_a-\delta_a) Q_{na} Z_n \quad \forall a \tag{13}$$

This constraint determines amount of units of each component send for disposal from dismantling center.

- Fabrication center

$$RI_a \leqslant \phi_a G_a \quad \forall a \tag{14}$$

$$RI_a \geqslant FDEM_a \quad \forall a \tag{15}$$

The amount of components transported to fabrication center from component fabrication center are determined and are not exceeding the demand of fabrication center.

- Manufacturing center

$$M_a = \eta_a G_a \quad \forall a \tag{16}$$

$$MDEM_a = \sum_n Q_{na} D_n \quad \forall a \tag{17}$$

The amount of components transported to manufacturing center from component fabrication.

- Spare market

$$RF_{af} - (G_a - M_a - RI_a)P_{af} \quad \forall a, f \tag{18}$$

$$\sum_f P_{af} - 1 \quad \forall a \tag{19}$$

After satisfying the demands at fabrication center and service center the remaining components are transported to spare market.

- Component Supplier

$$CS_a = MDEM_a - M_a \quad \forall a \tag{20}$$

The amount of components to be procured from the component supplier.

- Distribution center

$$\sum_j DC_{jn} = D_n \quad \forall n \tag{21}$$

$$\sum_c X_{cn} H_{jc} = DC_{jn} \quad \forall j, n \tag{22}$$

The amount of products transported to distribution center from manufacturing center and the amount of products transported from distribution center to different customer zones.

- Non Negativity restriction

$$B_{mn}, RX_{mn}, Z_n, O_n, V_a, G_a, DIS_a, M_a, RI_a DM_{jm}, D_n, CS_a, MDEM_a, DC_a \geq 0 \quad \forall m, n, j, a \tag{23}$$

- Binary variable

$$Y_{cl}, P_{af}, H_{jc} \in \{0, 1\} \quad \forall c, l, a, f, j \tag{24}$$

- Integer restriction

$$B_{mn}, RX_{mn}, Z_n, O_n, V_a, G_a, DIS_a, M_a, RI_a DM_{jm}, D_n, CS_a, MDEM_a, DC_a \geq 0 \quad \forall m, n, j, a \in Integer \tag{25}$$

***The optimization model has been applied to a case of a company manufacturing electronic products. The case study can be provided on request.*

References

[1] Pishvaee, M.R., Rabbani, M., and Torabi, S.A., 2011, *A robust optimization approach to closed-loop supply chain network design under uncertainty*, Applied Mathematical Modeling, 35 (2), 637649.

[2] Guide, V. D. R., Van Wassenhove, L. N. (2009), *OR FORUM - The evolution of closed-loop supply chain research*, Operations Research, 57(1), $10 - 18$.

Jyoti Darbari, Assistant Professor, Department of Mathematics, LADY SHRI RAM COLLEGE FOR WOMEN

E-mail address: jydbr@hotmail.com

Vernika Agarwal,Department of Operational Research, University of Delhi,

E-mail address: vernika.agarwal@gmail.com

PC Jha, Associate Professor, Department of Operational Research, University of Delhi,,

E-mail address: jhapc@yahoo.com

ROBE'S RESTRICTED PROBLEM OF 2+2 BODIES REVISITED

BHAVNEET KAUR

Abstract. In this problem, one of the primaries of mass m_1 is a Roche Ellipsoid filled with a homogeneous incompressible fluid of density ρ_1. The smaller primary of mass m_2 is an oblate body outside the Ellipsoid. The third and the fourth bodies (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the Ellipsoid, with the assumption that the mass and the radius of the third and the fourth body are infinitesimal. We assume that m_2 is describing a circle around m_1. The masses m_3 and m_4 mutually attract each other, do not influence the motions of m_1 and m_2 but are influenced by them. We have extended the Robe's Restricted three-body problem to $2+2$ body problem under the assumption that the fluid body assumes the shape of the Roche Ellipsoid ([2] Chandrashekhar(1987)). We have taken into consideration all the three components of the pressure field in deriving the expression for the buoyancy force viz (i) due to the own gravitational field of the fluid (ii) that originating in the attraction of m_2 (iii) that arising from the centrifugal force. In this paper, equilibrium solutions of m_3 and m_4 and their linear stability are analyzed. We have proved that there exist only six equilibrium solutions of the system. In a system where the primaries are considered as earth-moon and m_3, m_4 as submarines, the equilibrium solutions of m_3 and m_4 respectively when the displacement is given in the direction of x_1- axis or x_2- axis are unstable.

Introduction

In([5]), the author has investigated a new kind of restricted three-body problem in which one of the primaries of mass m_1 is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1. The smaller primary is a mass point m_2 outside the shell. The third body of mass m_3, supposed to be moving inside the shell, is a small solid sphere of density ρ_3, with the assumption that the mass and the radius of the third body are infinitesimal. He further assumed that the mass m_2 describes a Keplerian orbit around the mass m_1. We studied the equilibrium solutions of m_3 and analysed their linear stability.

In deriving the expression for the buoyancy force, Robe assumed that the pressure field of the fluid ρ_1 has spherical symmetry about the centre of the shell. He has taken into account one of the three components of the pressure field, that is, due to the own gravitational field of the fluid. In ([4]), Plastino investigated the effect of the remaining two components: (i) that originating in the attraction of m_2, (ii) that arising from the centrifugal force. They incorporated these two components of pressure field into the dynamics of the Robe model by considering that the fluid m_1 adopts the shape of a Roche ellipsoid.

Statement of the Problem and Equations of Motion

In this paper, one of the primaries of mass m_1 is a Roche Ellipsoid filled with homogeneous incompressible fluid of density ρ_1. The second primary of mass m_2 ($m_1 > m_2$) is a an oblate body outside the Ellipsoid. The third and the fourth body (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the Ellipsoid, with the assumption that the mass and radius of the third and the fourth body are infinitesimal. Let R be the distance between the centres of mass of m_1 and m_2 . We assume that m_2 describes a circular orbit of radius R around m_1 with constant angular velocity $\boldsymbol{\omega}$. The masses m_3 and m_4 mutually attract each other but do not influence the motions of m_1 and m_2.

As in the case of classical restricted problem ([6] Szebehely(1967)), we adopt a uniformly rotating coordinate system $Ox_1x_2x_3$ with origin of the coordinate system at the centre of the bigger primary, Ox_1 pointing towards m_2 and Ox_1x_2 being the orbital plane of m_2 around m_1. The coordinate system $Ox_1x_2x_3$ is as shown in the Figure 1. Let the synodic system of coordinates initially coincident with the inertial system rotate with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1. Let initially the principal axes of m_2 be parallel to the synodic axes and their axes of symmetry be perpendicular to the plane of motion. Since m_2 is revolving without rotation about m_1 with the same angular velocity as that of the synodic axes, the principal axes of m_2 will remain parallel to them throughout the motion.

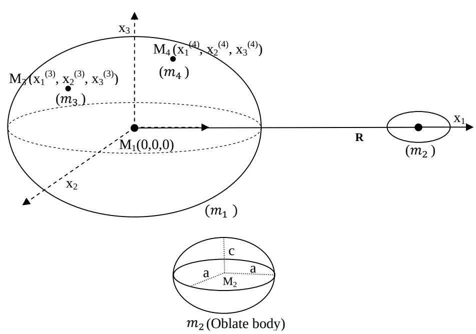


Figure 1. Geometry of the Robe's restricted problem of $\mathbf{2+2}$ bodies when the bigger primary $\mathbf{m_1}$ is a Roche Ellipsoid and the smaller primary an oblate body

The Equations of motion of m_3 and similarly of m_4 in the dimensionless cartesian coordinates are

$$\ddot{x}_1^{(i)} - 2\omega \dot{x}_2^{(i)} = V^{(i)}_{x_1^{(i)}}, \tag{1}$$

$$\ddot{x}_2^{(i)} + 2\omega \dot{x}_1^{(i)} = V^{(i)}_{x_2^{(i)}}, \tag{2}$$

$$\ddot{x}_3^{(i)} = V^{(i)}_{x_3^{(i)}}, \tag{3}$$

where

$$V^{(i)} = \frac{\mu_j}{R_{ij}} + D_i\Big[B_i + \frac{1}{2}\omega^2\left\{\left(x_1^{(i)}\right)^2 + \left(x_2^{(i)}\right)^2\right\} + \mu\left\{\left(x_1^{(i)}\right)^2 - \frac{1}{2}\left(x_2^{(i)}\right)^2 - \frac{1}{2}\left(x_3^{(i)}\right)^2\right\}$$
$$+ \frac{\mu A}{2}\left\{\left(6x_1^{(i)}\right)^2 - \frac{3}{2}\left(x_2^{(i)}\right)^2 - \frac{9}{2}\left(x_3^{(i)}\right)^2\right\} + \left(\frac{\mu + \frac{3}{2}\mu A}{\omega}\right)^2\Big].$$

$$\mu_j = \frac{m_j}{m_1 + m_2}, D_i = \left(1 - \frac{\rho_1}{\rho_i}\right) \qquad i, j = 3, 4; i \neq j. \tag{4}$$

$$B_i = \pi G \rho_1\left(I - A_1\left(x_1^{(i)}\right)^2 - A_2\left(x_2^{(i)}\right)^2 - A_3\left(x_3^{(i)}\right)^2\right)$$

The Equations of motion of m_3 and m_4 can be rewritten as

$$\ddot{x}_1^{(i)} - 2\omega\dot{x}_2^{(i)} = -\mu_j\frac{\left(x_1^{(i)} - x_1^{(j)}\right)}{R_{ij}^3} + D_i\left(\omega^2 + 2\mu - C_1 + 6\mu A\right)x_1^{(i)}, \tag{5}$$

$$\ddot{x}_2^{(i)} + 2\omega\dot{x}_1^{(i)} = -\mu_j\frac{\left(x_2^{(i)} - x_2^{(j)}\right)}{R_{ij}^3} + D_i\left(\omega^2 - \mu - C_2 - \frac{3}{2}\mu A\right)x_2^{(i)}, \tag{6}$$

$$\ddot{x}_3^{(i)} = -\mu_j\frac{\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^3} + D_i\left(-\mu - C_3 - \frac{9}{2}\mu A\right)x_3^{(i)}, \tag{7}$$

where

$$i, j = 3, 4; i \neq j \tag{8}$$

and

$$C_l = 2\left(\pi G\rho_1\right)A_l \tag{9}$$

$$= 2\left(\frac{\mu}{\mu^*}\right)A_l\;(l = 1, 2, 3) \tag{10}$$

$$A = \frac{a^2 - c^2}{5R^2}(\text{[3]Bhavneet and Rajiv(2014)}) \tag{11}$$

and

$$\mu^* = \frac{\mu}{\pi G\rho_1}. \tag{12}$$

Equilibrium Solutions

The equilibrium solutions of m_3 and m_4 are given by

$$V_{x_1^{(i)}}^{(i)} = V_{x_2^{(i)}}^{(i)} = V_{x_3^{(i)}}^{(i)} = 0 \quad (i = 3, 4).$$

i.e. ,

$$-\mu_j \frac{\left(x_1^{(i)} - x_1^{(j)}\right)}{R_{ij}^3} + D_i \left(\omega^2 + 2\mu - C_1 + 6\mu A\right) x_1^{(i)} = 0, \tag{13}$$

$$-\mu_j \frac{\left(x_2^{(i)} - x_2^{(j)}\right)}{R_{ij}^3} + D_i \left(\omega^2 - \mu - C_2 - \frac{3}{2}\mu A\right) x_2^{(i)} = 0, \tag{14}$$

$$-\mu_j \frac{\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^3} + D_i \left(-\mu - C_3 - \frac{9}{2}\mu A\right) x_3^{(i)} = 0. \tag{15}$$

with

$$i, j = 3, 4; i \neq j.$$

Equilibrium Solutions lying on x_1- axis
By inspection, we see that the Equations (14),(15) are satisfied with $x_2^{(i)}, x_3^{(i)}; i = 3, 4$ equal to zero. It remains to determine $x_1^{(i)}; i = 3, 4$ from the Equation (13) by putting $x_2^{(i)}, x_3^{(i)}; i = 3, 4$ equal to zero, we get

$$-\mu_4 \frac{\left(x_1^{(3)} - x_1^{(4)}\right)}{|x_1^{(3)} - x_1^{(4)}|^3} + \left\{D_3 \left(\omega^2 + 2\mu - C_1 + 6\mu A\right)\right\} x_1^{(3)} = 0. \tag{16}$$

$$-\mu_3 \frac{\left(x_1^{(4)} - x_1^{(3)}\right)}{|x_1^{(4)} - x_1^{(3)}|^3} + \left\{D_4 \left(\omega^2 + 2\mu - C_1 + 6\mu A\right)\right\} x_1^{(4)} = 0. \tag{17}$$

Multiplying the Equation (16) by μ_3 and (17) by μ_4 and adding, we get

$$x_1^{(4)} = -\lambda x_1^{(3)} \tag{18}$$

where

$$\lambda = \frac{D_3 \mu_3}{D_4 \mu_4} \tag{19}$$

Substituting in the Equation (16), we get

$$x_1^{(3)} = \pm \left[\frac{\mu_4}{\left\{D_3 \left(\omega^2 + 2\mu - C_1 + 6\mu A\right)\right\} (1 + \lambda)^2}\right]^{\frac{1}{3}}, \tag{20}$$

and

$$x_1^{(4)} = \mp \lambda \left[\frac{\mu_4}{\left\{D_3 \left(\omega^2 + 2\mu - C_1 + 6\mu A\right)\right\} (1 + \lambda)^2}\right]^{\frac{1}{3}}. \tag{21}$$

Hence $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ are the equilibrium solutions for m_3 and m_4 respectively provided they lie within the Roche's Ellipsoid. It may be noted that there are two values of $x_1^{(3)}$ and two values of $x_1^{(4)}$. We denote these values by $\xi_1^{(3)}, \xi_2^{(3)}$ and $\xi_1^{(4)}, \xi_2^{(4)}$ respectively. Therefore, the positions of equilibrium for the system are $\left(\xi_1^{(3)}, 0, 0\right), \left(\xi_1^{(4)}, 0, 0\right)$ and $\left(\xi_2^{(3)}, 0, 0\right), \left(\xi_2^{(4)}, 0, 0\right)$. We observe that substituting the Equation (18) in Equation

(17) yields the same equilibrium solutions. The equilibrium solutions of m_3 lying on x_1- axis are shown in Figure 2. Similarily, we find the equilibrium solutions lying on x_2 and x_3- axis.
There exist two equilibrium solutions of the system lying each on x_1- axis or x_2- axis or x_3- axis. Hence, there exist six equilibrium solutions of the system, provided they lie within Roche Ellipsoid. We observe that there are no other equilibrium solutions except these six.

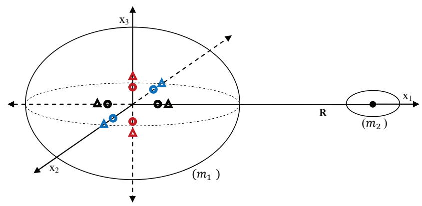


FIGURE 2. Location of Equilibrium Solutions of the Robe's Restricted Problem of 2+2 Bodies when the Bigger Primary is a Roche Ellipsoid and the Smaller Primary is an Oblate Body. Circles denote the positions of $\mathbf{m_3}$ and triangles denote the positions of $\mathbf{m_4}$.

STABILITY OF EQUILIBRIUM SOLUTIONS

Stability of Equilibrium Solutions lying on x_1- axis

Let the equilibrium solution $\left(x_1^{(3)},0,0\right)$ and $\left(x_1^{(4)},0,0\right)$ of m_3 and m_4 be displaced to $\left(x_1^{(3)}+\alpha_1^{(3)},\alpha_2^{(3)},\alpha_3^{(3)}\right)$ and $\left(x_1^{(4)}+\alpha_1^{(4)},\alpha_2^{(4)},\alpha_3^{(4)}\right)$.
The variational equations of m_3 and m_4 are

$$\ddot{\alpha}_1^{(i)}-2\omega\dot{\alpha}_2^{(i)}=\alpha_1^{(i)}p_1^{(i)}\left(\frac{3+\lambda}{1+\lambda}\right) \tag{22}$$

$$\ddot{\alpha}_2^{(i)}+2\omega\dot{\alpha}_1^{(i)}=\alpha_2^{(i)}\left(\frac{-p_1^{(i)}}{1+\lambda}+p_2^{(i)}\right) \tag{23}$$

$$\ddot{\alpha}_3^{(i)}=\alpha_3^{(i)}\left(\frac{-p_1^{(i)}}{1+\lambda}+p_3^{(i)}\right) \qquad i=3,4 \tag{24}$$

where

$$\lambda=\frac{D_3\mu_3}{D_4\mu_4},$$
$$p_1^{(i)}=D_i\left(1+2\mu-C_1+A\left(6\mu+\frac{3}{2}\right)\right)$$
$$p_2^{(i)}=D_i\left(1-\mu-C_2+\frac{3}{2}A\left(1-\mu\right)\right)$$
$$p_3^{(i)}=D_i\left(-\mu-C_3-\frac{9}{2}\mu A\right).$$

From the Equation (24), we ascertain that the motion of m_3 and m_4 parallel to x_3 axis is stable when

$$\frac{p_1^{(i)}}{p_3^{(i)}} > 1+\lambda \quad i=3,4 \ \text{ in case } \rho_1 > \rho_3, \rho_1 > \rho_4 \tag{25}$$

and

$$\frac{p_1^{(i)}}{p_3^{(i)}} < 1+\lambda \quad i=3,4 \ \text{ in case } \rho_1 < \rho_3, \rho_1 < \rho_4 \tag{26}$$

The remaining Equations (22) and (23) admit solutions of the form $\alpha_i^{(j)} = A_i^{(j)} e^{L_j t}, i = 1,2,; j=3,4$.
The characteristic equations of m_3 and m_4 are given by

$$L_i^4 + L_i^2 \left(4\omega^2 - \left(Q_1^{(i)} + Q_2^{(i)}\right)\right) + Q_1^{(i)} Q_2^{(i)} = 0 \quad i=3,4 \tag{27}$$

where

$$Q_1^{(i)} = p_1^{(i)} \left(\frac{3+\lambda}{1+\lambda}\right)$$
$$Q_2^{(i)} = p_2^{(i)} - \frac{p_1^{(i)}}{1+\lambda}.$$

Let $\Lambda_i = L_i^2$, we obtain,

$$\Lambda_i^2 + \Lambda_i \left(4\omega^2 - \left(Q_1^{(i)} + Q_2^{(i)}\right)\right) + Q_1^{(i)} Q_2^{(i)} = 0 \quad i=3,4. \tag{28}$$

Let the roots of the Equation (28) be $S_1^{(i)}$, $S_2^{(i)}$. Then, $S_1^{(i)} + S_2^{(i)} = \left(Q_1^{(i)} + Q_2^{(i)}\right) - 4\omega^2$.
$S_1^{(i)} S_2^{(i)} = Q_1^{(i)} Q_2^{(i)}$
Discriminant Δ of the Equation (28) is:

$$\begin{aligned}\Delta &= 16\omega^4 + \left(Q_1^{(i)} - Q_2^{(i)}\right)^2 - 8\omega^2 \left(Q_1^{(i)} + Q_2^{(i)}\right)\\ &= \left(Q_1^{(i)} - Q_2^{(i)}\right)^2 + 8\omega^2 \left(2\omega^2 - \left(Q_1^{(i)} + Q_2^{(i)}\right)\right)\end{aligned}$$

Case I: $\rho_1 < \rho_3$, $\rho_1 < \rho_4$.
In this case $D_i > 0, \lambda > 0, p_1^{(i)} < 0, p_2^{(i)} < 0, \quad (i=3,4)$ as $C_1 > 1+2\mu$ and $C_2 > 1-\mu$.
Thus $Q_1^{(i)} < 0$.
The equilibrium solutions $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ of m_3 and m_4 respectively when the displacement is given in the direction of x_1- axis or x_2- axis are stable if $S_1^{(i)}$ and $S_2^{(i)}$ are real and negative or we must have $S_1^{(i)} + S_2^{(i)} < 0$, $S_1^{(i)} S_2^{(i)} > 0$, $\Delta > 0$.
Now, $S_1^{(i)} S_2^{(i)} > 0$ if $Q_2^{(i)} < 0$ i.e. if,

$$\frac{p_1^{(i)}}{p_2^{(i)}} < 1+\lambda \quad i=3,4 \tag{29}$$

i.e ,

$$\frac{\left(1+2\mu-C_1+A\left(6\mu+\frac{3}{2}\right)\right)}{\left(1-\mu-C_2+\frac{3}{2}A\left(1-\mu\right)\right)}<1+\lambda \quad i=3,4. \tag{30}$$

This implies that $S_1^{(i)}+S_2^{(i)}<0, \Delta>0$. Hence, the equilibrium solutions of m_3 and m_4 are stable if the Equation (30) holds, else unstable.

Case II: $\rho_1>\rho_3, \rho_1>\rho_4$.

In this case $D_i<0,\ \lambda>0,\ p_1^{(i)}>0,\ p_2^{(i)}>0\ \ (i=3,4)$. Thus $Q_1^{(i)}>0$. The equilibrium solutions $\left(x_1^{(3)},0,0\right)$ and $\left(x_1^{(4)},0,0\right)$ of m_3 and m_4 respectively when the displacement is given in the direction of x_1- axis or x_2- axis are stable if $S_1^{(i)}+S_2^{(i)}<0,\ S_1^{(i)}S_2^{(i)}>0$, $\Delta>0$.

Now, $S_1^{(i)}S_2^{(i)}>0$ if $Q_2^{(i)}>0$. The region that is common to $Q_2^{(i)}>0$, $\left(Q_1^{(i)}+Q_2^{(i)}\right)<2\omega^2$ and $\Delta>0$ is the stable region. The shaded region in Figure 3 shows the region in which equilibrium solutions m_3 are stable for $\lambda=0.1$ and $A\approx10^{-8}$ where

$$Q_1^{(i)}+Q_2^{(i)}=p_1^{(i)}\left(\frac{2+\lambda}{1+\lambda}\right)+p_2^{(i)}$$

$$\Delta=\left(p_1^{(i)}\left(\frac{4+\lambda}{1+\lambda}\right)-p_2^{(i)}\right)^2+8\omega^2\left(2\omega^2-p_1^{(i)}\left(\frac{2+\lambda}{1+\lambda}\right)-p_2^{(i)}\right)$$

Similarly, we can study the stability of equilibrium solutions lying on x_2- axis and x_3- axis.

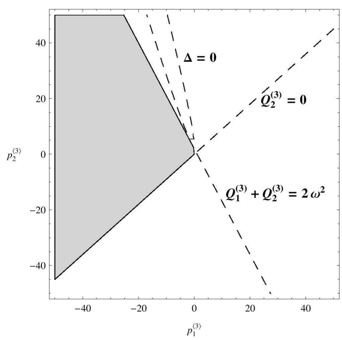


FIGURE 3. The shaded region corresponds to the region of stability for the equilibrium solutions of $\mathbf{m_3}$ lying on x_1 axis

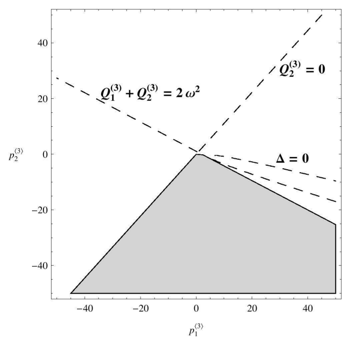


FIGURE 4. The shaded region corresponds to the region of stability for the equilibrium solutions of $\mathbf{m_3}$ lying on x_2 axis.

CONCLUSION

Celestial bodies in general are not spherical, rather they are oblate or axis symmetric bodies. It is therefore essential that we concentrate on primaries which are axis symmetric

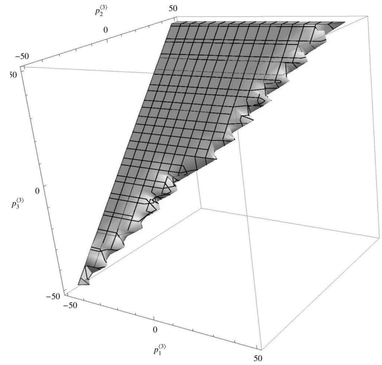


FIGURE 5. The shaded region corresponds to the region of stability for the equilibrium solutions of $\mathbf{m_3}$ lying on x_3 axis.

bodies and preferably on oblate bodies. We have studied the motion of two infinitesimal masses m_3 and m_4 supposedly moving inside m_1 in three dimensions, taking m_2 an oblate body and m_1 a Roche Ellipsoid. We have taken into consideration all the three components of the pressure field in deriving the expression for the buoyancy force viz, due to the own gravitational field of the fluid, that originating in the attraction of m_2 and that arising from the centrifugal force. We have proved that there exist only six equilibrium solutions of the system, provided they lie within the Roche Ellipsoid. We have also studied the stability of these equilibrium solutions.

Disclaimer: This article is a brief overview of the reserach work published in my paper 'Robe's Restricted Problem of 2+2 Bodies when the Bigger Primary is a Roche Ellipsoid and the Smaller Primary is an Oblate Body' in the journal Astrophysics and Space Science. It has been included in this journal to motivate students to do research in this field.

REFERENCES

[1] *Dirk Brouwer & Gerald Maurice Clemence*, G.M.: 1961 'Methods of Celestial Mechanics', Academic Press, New York and London.

[2] *Chandrashekhar*, S.: 1987 'Ellipsoidal Figures of Equilibrium'(Chapter 8), Dover Publication Inc, New York.

[3] *Kaur*, *Bhavneet* & *Aggarwal*, R.: 2014 'Robe's Restricted Problem of 2+2 Bodies when the Bigger Primary is a Roche Ellipsoid and the Smaller Primary is an Oblate Body', Astrophysics and Space Science 349,5769.

[4] *A.R.Plastino*& *A.Plastino*: 1995, 'Robe's Restricted Three-Body Problem Revisited', Celestial Mechanics and Dynamical Astronomy 61, 197-206.

[5] *Robe*, H.A.G.: 1977, 'A New Kind of Three-Body Problem', Celestial Mechanics 16, 343-351.

[6] *Szebehely*, V.S.: 1967 'Theory of orbits ', Academic Press, New York.

BHAVNEET KAUR, ASSISTANT PROFESSOR, MATHEMATICS DEPARTMENT, LADY SHRI RAM COLLEGE FOR WOMEN

***E-mail address*:** `bhavneet.lsr@gmail.com`

Extension of Course Contents

A great deal of learning happens beyond the formal coursework. This section hence, aims to provide a creative, fertile setting for productive research that goes beyond the confines of classroom, and precincts of syllabi. It strengthens and expands the existing knowledge and adds interests to the course and provides an experience of transformative learning.

DIFFERENT PATHS LEADING TO ONE TARGET

MONIKA SINGH

ABSTRACT. The present article highlights the beauty of mathematics in re-exploring a concept. Here we give 9 different proofs of the Euclid's theorem.

INTRODUCTION

Besides evolving new concepts and results, one of the inherent characteristic of mathematics is to search for different proofs, for different reasons, for an established result, and specially the popular ones, namely, the one like - **Euclid's Theorem**. This article contains as many as 9 different proofs of Euclids Theorem, given by mathematicians time to time using different techniques. For example, the proofs given by Euler, Goldbach, Mersenne, Whang and Thomas Stieltjes etc. including the classical ones given by Euclid himself are given here. In fact, an interesting thing to see would be that how one theorem can be proved just by using a simple technique/argument of mathematics and at the same time, it can also be proved by using sophisticated tools of mathematics !

In the article, any prerequisites required for a particular proof has been given at the appropriate place to make the article self contained.

Eulid's Theorem (statement): "Prime numbers are infinite".

PROOF BY EUCLID

Definition 1. A positive integer $p > 1$ is called *prime number*, or simply a *prime* if its only positive divisors are 1 and p itself. For example, 2,3,5,7,..., 17,..., 37,..., 89..... etc.

Theorem 1 (Fundamental Theorem of Arithmetic, see [2]). Every positive integer $n > 1$ can be expressed as the product of primes. Moreover, the representation is unique up to the order in which the factors occur.

Proof 1. We prove the theorem by contradiction. Suppose there are finitely many primes, say $p_1, p_2, ..., p_n$. Consider the positive integer $N = p_1 p_2 ... p_n + 1$.

Date: 24/03/2014.

If N is prime, then it means we have one more prime greater than all the primes $p_1,\ p_2,\ ...,p_n$, which is a contradiction to our assumption.

If N is not prime, then as $N > 1$, by Theorem 1, N should be divisible by some prime, say p. Clearly, this p must be one of the primes $p_1,\ p_2,\ ...,p_n$.

$$\Rightarrow p|(p_1p_2...p_n) \text{ and } p|N \Rightarrow p|(N - p_1p_2...p_n) \Rightarrow p|1,$$

which is a contradiction, since by definition a prime must be greater than 1.
Thus, the prime numbers can't be finite in counting, and hence, they are infinite in number.
□

Proof 2 (see [2]). On the contrary suppose there are finitely many primes, say $p_1,\ p_2,\ ...,p_n$, arranged in an ascending order,i.e., say, $p := p_n$ be the largest of these. Consider the positive integer $N = p\,! + 1$.

If N is prime, then it means we have one more prime greater than all the primes $p_1,\ p_2,\ ...,p_n$, which is a contradiction to our assumption.

If N is not prime, then as $N > 1$, by Theorem 1, N should be divisible by some prime, say q. Now since we have assumed only to be a finite number of primes, this q must be one of the primes $p_1,\ p_2,\ ...,p_n$, i.e., $1 < q \leq p$.

$$\Rightarrow q\,|\,p\,! \Rightarrow q\,|\,(N - p!) \Rightarrow q|1,$$

which is a contradiction, since by definition a prime must be greater than 1.
Thus, the prime numbers can't be finite in counting, and hence, they are infinite in number.
□

Proof by Braun

Proof (see [2]). On the contrary suppose there are finitely many primes, say $p_1,\ p_2,\ ...,p_n$. Take

$$N = p_2\,p_3\,...p_n + p_1\,p_3\,...p_n + ... + p_1\,p_2\,...p_{n-1}. \tag{1}$$

Clearly, $N > 1$. Therefore, by Theorem 1, there exists a prime p such that $p\,|\,N$. Clearly, this p must be one of the primes $p_1,\ p_2,\ ...,p_n$. I.e., $p = p_i,\ 1 \leq i \leq n$.

Thus, p will divide all the terms in the R.H.S. of (1) except the term $p_1\,p_2\,...p_{i-1}p_{i+1}\,...p_n$.

$$\begin{aligned}\Rightarrow p\,|\,\{N - (&p_2\,p_3\,...p_n + p_1\,p_3\,...p_n + p_1\,p_2\,...p_{i-2}p_i\,...p_n \\ &+ p_1\,p_2...\,p_i\,p_{i+2}\,...p_n + ... + p_1\,p_2\,...p_{i-1}\,p_i\,p_{i+1}...p_{n-1})\}\end{aligned}$$

$\Rightarrow p\,|p_1\,p_2\,...p_{i-1}\,p_{i+1}...p_n$, which is a contradiction. Hence our assumption goes wrong, and there are infinite number of primes. □

Proof by using Euler's Totient Function

Definition 2 (Euler ϕ-function). For $n \geq 1$, let $\phi(n)$ denote the number of positive integers not exceeding n that are relatively prime to n. For example, $\phi(1) = 1$, $\phi(2) = 1$, $\phi(3) = 2$ etc.

Observation. $\phi(p) = p - 1$, whenever p is prime.

Definition 3. A function $f : \mathbb{N} \to \mathbb{R}$ is said to be *a multiplicative function* if

$$f(mn) = f(m)f(n)$$

whenever $gcd(m,\, n) = 1,\ m, n\ \in \mathbb{N}$. For example, the constant function 1 and the identity function are trivially the multiplicative functions.

Theorem 2 (see [2]). The Euler ϕ-function is a multiplicative function.

Theorem 3 (see [2]). For $n > 2$, $\phi(n)$ is an even integer.

Proof ([2]). On the contrary suppose there are finitely many primes, say $p_1,\, p_2,\, ..., p_n$. Take $N = p_1 p_2 ... p_n$. **Claim:** If $1 < a \leq \mathbb{N}$, then $gcd(a,\, N) \neq 1$.

Since $a > 1$, by the Fundamental Theorem of Arithmetic a has a prime divisor, say q. Now since we have assumed only to be a finite number of primes, this q must be one of the primes $p_1,\, p_2,\, ..., p_n$. I.e., $q \,|\, N$. Therefore, $gcd(a,\, N) \geq q$, and it is so for all $1 < a \leq \mathbb{N}$. Hence $gcd(a,\, N) \neq 1$ for all $1 < a \leq \mathbb{N}$. Consequently, we can say that 1 is the only number less than N and relatively prime to it. I.e., $\phi(N) = 1$, which is a contradiction in view of Theorem 3. Hence there are infinite number of primes. □

Proof by Goldbach

Definition 4. A *Fermat number* is a positive integer of the form $F_n = 2^{2^n} + 1$, $n = 0, 1, 2, 3, ...$ Moreover, if F_n is prime, it is said to be *Fermat prime.* For example, $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$ and so on.

Observation. Clearly, the Fermat numbers are infinite in number.

Theorem 4 (see [2]). For Fermat numbers F_m and F_n, where $m \neq n$, $gcd(F_m,\, F_n) = 1$.

Proof ([2]). Since each Fermat number is greater than 1, by Fundamental theorem of Arithmetic, each Fermat number is divisible by a prime. Then in view of Theorem 4, all such primes should be distinct, otherwise there will exist at least one pair of Fermat numbers, say F_r, F_s such that $gcd(F_r,\, F_s) \neq 1$, which is a contradiction. Therefore, there are as many primes as there are Fermat numbers. Hence the number of primes is infinite. □

Proof by Stieltjes

Proof ([4]). On the contrary suppose there are finitely many primes, say $p_1, p_2, ..., p_r$. Take $K = p_1p_2...p_r$. Therefore K can be written as $K = mn$, where $m, n \geq 1$ and $\gcd(m, n) = 1$. Clearly, any of the primes p_i's $(1 \leq i \leq r)$ will divide exactly one of m and n, so that non of the p_i's divides $m + n$. But $m + n > 1$, so by the Fundamental Theorem of Arithmetic it should have at least one prime divisor. Thus, both the statements are contradictory to each other. Hence our assumption goes wrong, and there are infinite number of primes. □

Proof using Abstract Algebra

Theorem 5 (Lagrange's Theorem [3]). If G is a finite group and H a subgroup of G, then o(H) divides o(G).

Theorem 6 (see [3]). Let G be a group and $a \in G$, then $o(a)|o(G)$.

Definition 5 (see [2]). The numbers of the form $M_n = 2^n - 1$, $n \geq 1$ are called *Mersenne numbers.*

Note. Let q be a prime, then we define Z_q^* the group of integers modulo q as - $Z_q^* := \{1, 2, ..., q-1\}$. Here the binary operation is $a \bigotimes_q b$. Clearly, $o(Z_q^*) = q - 1$.

Proof ([4]). Let p be an arbitrary prime. Consider $M_p = 2^p - 1$. Since $M_p > 1$, by the Fundamental Theorem of Arithmetic there exists some prime, say q, such that $q|M_p$.
Claim: $q > p$.
Now we have $q \,|\, (2^p - 1) \;\Rightarrow\; 2^p \;\equiv 1 \mod q$.
Since p is prime, $o(2) = p$ otherwise, $o(2)|p$, which is not possible.
As $2 \in Z_q^* = G$, by Theorem 6 we have $o(2)|o(G) \;\Rightarrow\; p|(q-1) \;\Rightarrow p < q$.
It means for every prime p, there exists a greater one. Hence the number of primes is on and on, i.e., infinite. □

Proof using Calculus

Definition 6. For $x \in \mathbb{R}^+$, we define the function $\pi(x)$ by

$$\pi(x) := \sharp\{p \leq x : p \in \mathbb{P}\},$$

where $\mathbb{P}$ denotes the collection of all primes. I.e., the function $\pi(x)$ gives the number of primes that are less than a given positive real number x.

Definition 7. For any real number x, the function $[x]$ called as the *greatest integer function* takes the value equal to the greatest integer less than or equal to x. I.e., $[x]$ is the unique integer satisfying $x - 1 < [x] \leq x$.

Proof. We claim that: $\pi(x)$ is an unbounded function of x.

Let $\mathbb{P} = \{p_1, p_2, p_3, ...\}$ be the collection of all primes arranged in ascending order. Consider the natural logarithm $\log(x)$ defined as $\log(x) := \int_1^x \frac{1}{t}\, dt$. Observe that if consider the partition $P = \{1, 2, 3, ..., n+1\}$ of [1, n+1], where $n = [x]$, then the area under the graph of the function $f(t) = 1/t$ for $t \in [1, x]$, would be less than the upper Darboux sum of f taken on P. I.e.,

$$\begin{aligned}\log(x) &\leq 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{n}\\ &\leq \sum \frac{1}{m},\end{aligned}$$

where the above sum extends over all $m \in \mathbb{N}$ which have **only** prime divisors $p \leq x$. Now, since by the Fundamental Theorem of Arithmetic every such m can be written uniquely as the product of the form $\bigcup\limits_{p \leq x} p^{k_p}$, one may see that the last sum is equal to

$$\prod_{p \leq x \,:\, p \in \mathbb{P}} \Big(\sum_{k \geq 0} \frac{1}{p^k}\Big).$$

The inner sum is a geometric series with common ratio $\frac{1}{p} < 1$, hence

$$\begin{aligned}\log(x) &\leq \prod_{p \leq x \,:\, p \in \mathbb{P}} \frac{1}{1 - 1/p}\\ &= \prod_{p \leq x \,:\, p \in \mathbb{P}} \frac{p}{p-1} = \prod_{k=1}^{\pi(x)} \frac{p_k}{p_k - 1}.\end{aligned}$$

Now clearly, $p_k \geq k+1$, and thus

$$\frac{p_k}{p_k - 1} = 1 + \frac{1}{p_k - 1} \leq 1 + \frac{1}{k} = \frac{k+1}{k},$$

and therefore

$$\log(x) \leq \prod_{k=1}^{\pi(x)} \frac{k+1}{k} = \pi(x) + 1.$$

And now since $\log(x)$ is unbounded, so would be $\pi(x)$. So the claim is made. Hence the number of primes are infinite. □

Proof by Whang

Theorem 7 (Ratio Test [6]). If $\sum a_n$ is a positive term series and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$. Then

(i) the series $\sum a_n$ converges if $l < 1$;
(ii) the series $\sum a_n$ diverges if $l > 1$;
(iiI) the test fails if $l = 1$.

Note. One may easily verify that by the Ratio test, the series $\sum\limits_{n=1}^{\infty} \frac{a^n}{n!}$, $a > 0$ is convergent.

Theorem 8 (see [6]). If $\sum a_n,\ a_n > 0$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

Remark. The above Note and Theorem 8 combined together give: $\lim_{n\to\infty} \frac{a^n}{n!} = 0,\ a > 0$.

Theorem 9 (see [2]). If n is a positive integer and p is a prime, then the exponent of the highest power of p that divides $n!$ is $\sum_{k=1}^{\infty} \left[\frac{n}{p^k}\right]$, where the infinite series remains alive only for a finite number of terms, since $\left[\frac{n}{p^k}\right] = 0$ for $p^k > n$.

Proof. Let k be any positive integer greater than 1. Then by the Fundamental Theorem of Arithmetic and Theorem 9, we have

$$k\,! = \prod_p p^{g(p,\,k)}, \text{ where p is prime}$$

and

$$\begin{aligned} g(p,\,k) &= \sum_{j=1}^{\infty} \left[\frac{k}{p^j}\right] = \left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \ldots \\ &< \frac{k}{p} + \frac{k}{p^2} + \ldots = \frac{k}{p}\Big(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\Big) \\ &= \frac{k}{p}\Big(\frac{1}{1-1/p}\Big) = \frac{k}{p-1} \le k. \end{aligned}$$

Thus we have

$$k\,! = \prod_p p^{g(p,\,k)} \le \prod_p p^k.$$

Or,

$$1 \le \frac{\prod_p p^k}{k!} = \frac{\Big(\prod_p p\Big)^k}{k!}. \tag{2}$$

Now on the contrary if the number of primes are finite then $\prod_p p = a(\text{say})$ will be finite, so that $\lim_{k\to\infty} \frac{a^k}{k!} = 0$ by the above Remark. But this is not possible in view of the inequality (2). Hence our assumption goes wrong, and the primes are infinite. □

The rationale behind such an article is to create a motivation for the students to explore into mathematics with different mind sets. In search of different proofs of the Euclid's theorem I could find more than a dozen of proofs, but here I have given only those one by which an undergraduate student can get an essence of proving a result by using the techniques based upon the different concepts of mathematics, viz. number theory, algebra, calculus and analysis etc. This is basically a composed article and the author claims no originality for the contents.

References

[1] F. Saidak, *A new proof of Euclid's theorem*, Amer. Math. Monthly, 113(2006).

[2] David M. Burton, *Elementary Number Theory* (6e), Tata McGraw Hill, Indian print, 2007.

[3] Joseph A. Gallian, *Contemporary Abstract Algebra* (4e), Narosa Publishing House, New Delhi, 1999.

[4] I. Niven, H. S. Zukerman and H. L. Montgometry , *An Introduction to the Theory of Numbers* (5e), John Wiley and Sons Inc., India.

[5] N. Robbins, *Begining Number Theory* (2e), Narosa Publishing House, New Delhi, 2007.

[6] K. A. Ross, *Elementary Analysis:The Theory of Calculus*, Undergrauate Texts in Mathematics, Springer (SIE), Indian print, 2004.

Faculty Member, Mathematics Department, LADY SHRI RAM COLLEGE FOR WOMEN
E-mail address: monikasingh@lsr.du.ac.in

EUCLIDEAN GEOMETRY AND THE PARALLEL POSTULATE

KRITIBHA RAI

Abstract. This paper discusses Euclids axioms and the concept of Euclidean geometry. It will focus on the fifth axiom also called the parallel postulate. We will be talking about how the fifth postulate caused a lot of confusion in the early years. Many mathematicians tried to prove the parallel postulates from the existing four axioms. However, on expanding the picture and moving to a space, this method of proving the fifth axiom from the first four was wrong. To see how the fifth axiom does not follow from the first four axioms, we will be extending normal geometry to spherical and hyperbolic geometry.

Introduction

The geometry most of us are familiar with is the geometry of Euclid, but at the research level the word geometry has a much broader definition. Today's geometers do not spend much of their time with a ruler and compass. It is more dependent on the way we visualize things and how abstract these visualizations are.

Euclidean Geometry

To begin with let us define line and line segments. Like in a normal convention a line is used for a line that extends indefinitely in both directions, while a line segment is a line with two end-points.

The Euclids axioms are as follows:-

- Any two points can be joined by exactly one line segment.
- Any line segment can be extended to exactly one line.
- Given any point P and length r, there is a circle of radius r with P as its centre.
- Any two right angles are congruent.
- If a straight line intersects two straight lines L and M, and if the interior angles on one side of N adds up to less than 180 degree, then the lines L and M intersects on that side of N.

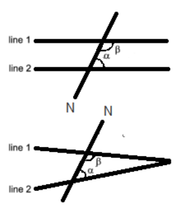


Figure 1

The 5^{th} axiom is illustrated in Figure 1. Where Line 1 is L and Line 2 is M

This 5^{th} axiom is equivalent to the so called Parallel Postulate, which states that given any line L and any point x not lying on L, there is exactly one line M that passes through x and never meets L. Euclid used these five axioms to build the whole of geometry as it was then understood. For example: The proof of a well known result that the angles of a triangle add up to 180 degrees is drawn from the 5^{th} axiom.

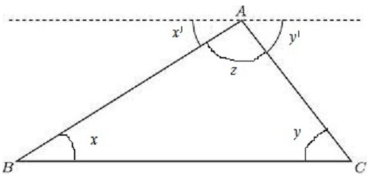


Figure 2

Referring to Figure 2, clearly by 5^{th} axiom, the alternate angles i.e. y and y' are equal also x and x' are equal. Otherwise the dotted line passing through A would meet the extended BC line. Also, since the dotted line is a straight line so the sum x', z, y' will be 180 and hence $x + y + z = 180$ as required.

What does this argument tell us about everyday life? An obvious conclusion that is evident is that if you take three points A, B, C in space and draw straight lines connecting them to form a triangle then the angle of this triangle will add up to 180 degrees. Further a simple experiment will confirm this: Just draw a triangle on a piece of paper and cut it out as neatly as you can, tear it into three pieces containing one corner each, place the corners together, and observe that the angles do indeed form a straight line.

Now, if you are convinced that there cannot exist a triangle whose angles do not add up to be exactly 180 degrees then you are moving towards understanding this topic! Well this was exactly the conclusion drawn by everybody from *Euclid* in $300BC$ to $Immanuel\ Kant$ at the end of the 18^{th} century.

However they were mistaken, about thirty years later the great mathematician $Carl$ $Friedrich\ Gauss$ could conceive of such a triangle in his famous experiment, where he tried to measure the angles of a triangle formed by the mountain peaks of Hohenhagen, Iselberg and Brocken in the Kingdom of Hanover. This experiment was inconclusive because of the difficulty of measuring the angles but this gave a new line of thought. Thus the truth of Euclid's axioms was questioned.

The Parallel Postulate

Out of all the axioms the most dubious was the 5^{th} axiom. It is more complicated than the other axioms and involves the infinite in a fundamental way. Simply put, is it not curious, when one proves that the angles of a triangle add up to 180 degrees, that the proof should depend on what happens in the outermost reaches of space?

Let us examine the parallel postulate more closely, and try to understand why it feels so obviously true. We think of two main arguments: the existence of a line parallel to a given line and the uniqueness of this line.

- Existence: Given a straight line C and a point x not on it, we can find a line passing through x parallel to C. Consider a point y on the same side of C as x and at the same distance from C. Join x and y by a line segment (axiom 1) and then extend this line segment to a full line A (axiom 2). Then A will never meet C and hence A is the required line.
- Uniqueness: Join A and C by evenly spread perpendicular line segments (Figure 3) with one of these segemnts through x. Now suppose that K is another line through x. On one side of x, the line K must lie between A and C, so it meets the next line segment at u. Say u is 1% of the way along the line segment from A to C. Then K will meet the next line segment 2% of the way along and so on. Thus, after 100 segements, K will have met C. Therefore proving the uniqueness of A.

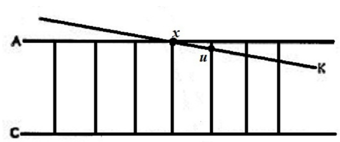


Figure 3

Finally, here is an argument that appears to show both the existence and uniqueness of a line parallel to C through a given point.

Notice that what we just did was that we tried to prove the parallel postulate and that is exactly what most of the mathematicians before the 19^{th} century tried to do. What they wanted was to deduce the parallel postulate from the other four axioms, thus showing that it was dispensable. However this was not possible because the arguments that we just gave and others like them contain many hidden assumptions which, when made explicit, are not obvious consequences of Euclid's first four axioms. Though plausible, they are no more plausible than the parallel postulate itself, thus bringing us back to square one.

Spherical Geometry

A valid way to bring these hidden assumptions out into the open is by examining the above arguments in a different context. In particular we choose the surface of a sphere, and show that the parallel postulate is not true here.

However, this might sound absurd as the the surface of sphere contains no straight lines at all. But if we simply apply the idea of reinterpretation i.e. if we redefine straight line, such that the surface of sphere does contain straight lines.

The most natural way is, a line segment from x to y is the shortest path from x to y that lies entirely within the surface of the sphere. You can imagine x and y to be two cities and the line segment as the shortest route that an aeroplane could take. Such a path will be a part of a 'great circle'. See Figure 4. Given the way we have defined a line segment, these 'great circles' make a good definition of a straight line in a spherical space.

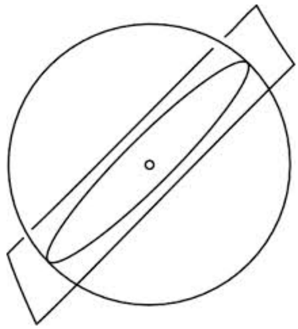


Figure 4

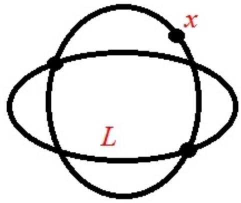


Figure 5

Clearly, if we adopt these definitions then parallel postulate is certainly false. For example, let L be the earth's equator and x be a point in the northern hemisphere. It is not hard to see that any great circle through x will lie half in the northern hemisphere and half in the southern hemisphere, crossing the equator at two points that are exactly opposite each other, see Figure 5. In otherwords there is no line through x that does not meet L. Also the above mentioned two arguments do not hold true for spherical geometry. Clearly the geometry which we think we are doing does not include possiblities of the spherical space. Euclid's first four axioms were devised to describe the geometry of an infinite, flat, two-dimensional space, but we are not obliged to interpret them that way, unless, of course this flatness follows from the axioms. Thus if we could somehow reinterpret these axioms by giving new definitions to phrases like lines and line segments and show that the first four axioms are true while the parallel postulate does not hold, then we are successful and we can say that the parallel postulate does not follow from the other axioms.

However, spherical geometry cannot be used as the 'reinterpretation' because even though the parallel postulate is false, not all the other first four Euclid's axioms are true in sphere. For example, a sphere does not contain circles of arbitrarily large radius, so axiom 3 fails, and there is not just one shortest route from the north pole to the south pole, so axiom 1 is also not true. Hence, although spherical geometry helped us understand the defects of certain attempted proofs of the parallel postulates, it still leaves open the possibility that some other proof might work. Hence we shall learn about another reinterpretation, called Hyperbolic Geometry. Here the parallel postulate will be false but the first four axioms holds true.

Hyperbolic Geometry

There are several equivalent ways of describing hyperbolic geometry; the one that we are going to use is the disc model, which was discovered by the great mathematician *Henri Poincaré*.

Since I can only provide limited information in this paper, we will discuss some of the main features of Hyperbolic Geometry and see what it tells us about the parallel postulate.

Understanding the disc model is more complicated than understanding the spherical geometry because we reinterpret not only 'lines' and 'line segments' but also the idea of distance.

Figure 6 shows a tessellation of the hyperbolic disc by regular pentagons. This statement needs explanation, since it is untrue if we think of distance in the usual way. However, distances in a hyperbolic disc are not defined in the usual way, and become larger, relative to normal distance, as we approach the boundary. Thus, it becomes clear why the pentagon (look at the one marked with an asterisk) appears to have one side longer than all the others. The other sides look shorter, but hyperbolic distance is defined in such a way that this apparent shortness is exactly compensated for by their being closer to the edge.

Before, moving on to discussing the definition of lines in hyperbolic disc, we will discuss about the Mercator's projection. A general map of the earth is flat but in actual earth is almost spherical and so distances in the map is necessarily distorted. Mercator's projection is one such way of carrying out the distortions. It is because of this that the countries near the pole appear to be much larger than they really are. The nearer you are to the top or bottom of such a map, the smaller the distances are, compared with what they appear to be.

As we saw earlier, when we approach the edge of the hyperbolic disc, distances become larger compared with how they look. As a result of this, the shortest path between two points has a tendency to deviate towards the centre of the disc. This means that it is not a straight line in the usual sense. It turns out that a hyperbolic straight line, that is, the shortest path from the point of view of hyperbolic geometry, is the arc of a circle that meets the boundary of the main circle at right angles (Figure 7)

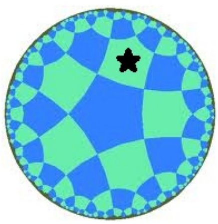


Figure 6: A tessellation of the hyperbolic disc by regular pentagons

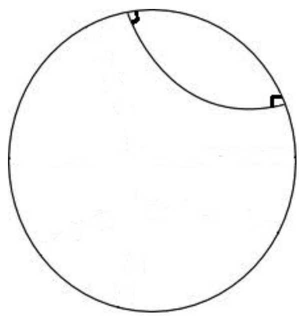


Figure 7

Referring to Figure 6 again we see that the edges of the pentagons, though they do not appear straight, are in fact hyperbolic line segments since they can be extended to hyperbolic lines. Thus, just like Mercator's projection, the disc model is a distorting 'map' of actual hyperbolic geometry.

Another very important property of hyperbolic geometry is that it satisfies all the first four Euclid's axioms. For example, any two points can be joined by exactly one hyperbolic straight line segment. Also, it may seem like we cannot find a circle of larger radius about any given point, but since the distances become larger near the edge of the disc and so if a hyperbolic circle almost brushes the edge, then its radius will be very large. See figure 8

Finally, we shall show that the parallel postulate is false for hyperbolic geometry. This can be seen in Figure 9, where lines M and N meets at a point x not lying on L and none of these two lines meet L, hence the uniqueness fails and so the parallel postulate doesn't hold true.

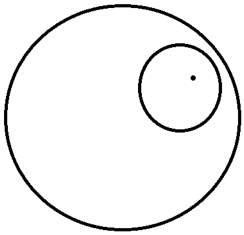


Figure 8: Hyperbolic circle happens to look like ordinary circles, but their centres are not where one expects them to be.

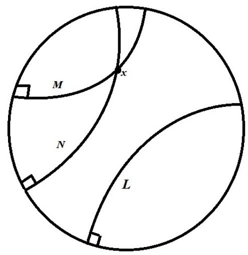


Figure 9

Conclusion

Clearly, geometry as a whole is very abstract. All the five Euclid's axioms are valid only in Euclidean geometry. In case of hyperbolic geometry, the first four axioms are valid but the parallel postulate doesn't hold true. This again removes the confusion that most of the mathematicians in the early 18th and 19th century had, and why they were not successful in deducing the parallel postulate from the other four axioms.

References

[1] Timothy Gowers, *Mathematics, a very short introduction*, Oxford University Press Inc., New York, 2002.
[2] www.britannica.com/EBchecked/topic/194901/Euclidean-geometryl
[3] www.wikipedia.org
[4] http://mathworld.wolfram.com/Non-EuclideanGeometry.html

KRITIBHA RAI, B.Sc.(H) MATHEMATICS, 6th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN

E-mail address: raikritibha5@gmail.com

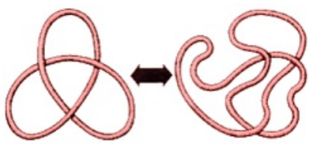
KNOT THEORY: KNOT INVARIANTS

URVASHI NEGI AND GARIMA YADAV

Abstract. In topology, knot theory is the study of mathematical knots. A knot is an embedding of a circle in 3-dimensional Euclidian Space, R^3. Two mathematical knots are equivalent if one can be transformed into the other via a deformation of R^3 upon itself and these transformations correspond to manipulations of a knotted string that do not involve cutting the string or passing the string through itself. The polynomial knot invariants are the most successful ways to tell knots apart.

INTRODUCTION

Two knots K_1, K_2 belonging to R^3 are ambient isotopic (isotopic) if and only if there exists an isotopy h from $R^3 \times [0,1]$ to R^3 s.t. h(K_1,0) = $h_0(K_1)$ =K_1, and h(K_1,1) =$h_1(K_1)$ =K_2. We denote this as K_1 ambient isotopic K_2 or K_1 isotopic K_2. Given below is an example of two equivalent knots.



Equivalent Knots

A link invariant is a function from the set of links to some other set whose value depends only on the equivalence class of the link. Any representative from the class can be chosen to calculate the invariant. There is no restriction on the kind of objects in the target space (i.e. integers, polynomials, matrices or groups). We will be discussing the following invariants of knots/links :

- Numeric Invariants: crossing number, unknotting number
- Colorability
- Polynomial Invariants (Alexander, Jones, HOMFLY polynomials)

NUMERIC INVARIANTS

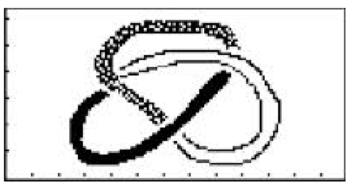
- The *crossing number* of a knot K, denoted by C(K) is the least number of crossings that occur in any diagram of the knot.
- A knot K has *unknotting number 'n'* (denoted by $u(K) = n$) if there exists a diagram of K such that changing n crossings in the diagram turns the knot into the unknot and there is no other diagram such that fewer changes turn it into the unknot.

$Trefoil$
$C(trefoil) = 3$ and $u(trefoil) = 1$

$Cinquefoil$
$C(cinquefoil) = 5$ and $u(cinquefoil) = 2$

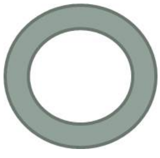
COLORABILITY

A diagram of a knot is colorable if and only if each arc can be drawn using one of the 3 colors such that at each crossing either 3 different colors come together or the same color comes together. Also, at least 2 of the colors must be used.
Trefoil is colorable.

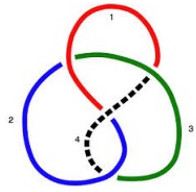


Colorability of $trefoil$ knot

Unknot is not colorable because we use only one color to draw the unknot.
Figure 8 knot is not colorable because there is a crossing for which two different colors meet.



Unknot is not colorable



$Figure$ 8 knot is not colorable

But colorability is not a complete invariant. Figure 8 and unknot are not colorable, so colorability cannot be used to show that Figure 8 knot is different from the unknot.

POLYNOMIAL INVARIANTS

The different knot polynomials are:

- **Alexander Polynomial (1928)**: It distinguishes all knots of 8 crossings or fewer, but it does not distinguish a knot from its mirror image (i.e. amphicheiral knots).
- **Jones polynomial (1984)**: It distinguishes all knots of 10 crossings or fewer, a knot from its mirror image, but it does not distinguish mutant knots.
- **HOMFLY polynomial (1985/87)**: It is a generalization of both Alexander and Jones polynomials. It is named after its inventors Hoste, Ocneanu, Millet, Freyd, Lickorish, Yetter (independently also Prztycki and Traczyk discovered the same polynomial). It does not distinguish mutant knots.

Presently, there is no complete polynomial invariant for knots!

ALEXANDER POLYNOMIAL

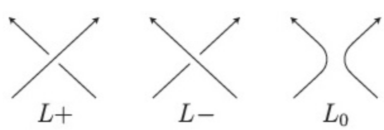
Alexander polynomial can be computed in several ways:

- **Alexander's combinatorially method (1928)**: It uses the diagram of the knots, and the Reidemeister moves (this method was presented at least 4 times as part of the work concerning the DK9 project).
- **Fox's method (1963)**: It uses a representation of the fundamental group of the complement of the knot. It was also mentioned in Alexander's original paper in his "Miscellaneous" section, but Fox's description is more detailed.
- **Conway's skein relation (1969)**: It uses skein relation, some special equations that connect the crossings of different knot diagrams. It was also mentioned in Alexander's original paper, but Conway's presentation is clearer and thus it paved the discovery for the Jones polynomial 15 years later.

JONES POLYNOMIAL

The Jones polynomial computed using the following rules is an invariant for knots:

- **Rule 1**: $V_{unknot}(t) = 1$
- **Rule 2**: Suppose that 3 knot/links differ at the arcs of one crossing as shown below:

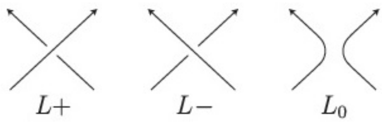


Then, $t^{-1} * V(L_+) - t * V(L_-) - (t^{1/2} - t^{-1/2}) * V(L_0) = 0$
For example, $V_{trefoil}(t) = -t^4 + t^3 + t$

HOMFLY POLYNOMIAL

The HOMFLY polynomial computed with the following rules is an invariant for knots:

- **Rule 1**: $P_{unknot}(l, m) = 1$
- **Rule 2**: Suppose that 3 knots/links differ at the arcs of one crossing as shown below:



Then, $l * P(L_+) + l^{-1} * P(L_-) + m * P(L_0) = 0$

Conclusion

The theory of knots is an exciting field of mathematics. Although computers may help us in distinguishing between knots, the fundamental problem of comparing knots remains open. We still have no complete invariant. It has concrete applications in the study of enzymes acting on DNA strands and a wide variety of applications in Biochemistry.

References

[1] V.O. Manturov: Knot theory .
[2] V. F. R. JONES: A polynomial invariant for knots via von Neumann algebras.
[3] www.wikipedia.org

URVASHI NEGI, GARIMA YADAV, B.Sc.(H) MATHEMATICS, 6th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN

E-mail address: urvashi.negi93@gmail.com, garima.y27@gmail.com

Interdisciplinary Aspects of Mathematics

Mathematics is just not a classroom discipline but a tool for organizing and understanding various concepts and applications. This section covers topics that delve into other disciplines, integrating the mode of thinking and knowledge of the respective discipline with Mathematics. The section hence highlights the cosmic scope of Mathematics, leveraging its amalgamation with other disciplines.

GAME THEORY AND BUSINESS STRATEGIES

NUPUR SOOD

Abstract. This paper discusses the application of game theory in forming business strategies. It will talk about some basic concepts of game theory such as the types of games, Prisoners Dilemma and Nash Equilibrium, further applying it to business situations such as entry and exit of firms in a market and price war. We shall also briefly discuss the recent news of social giant Facebook buying WhatsApp and the involvement of Game Theory in its decision making.

What is Game Theory?

A game consists of three components:

- Set of players.
- Set of available strategies for each player.
- Set of payoffs to each player for each possible configuration of strategies.

Game Theory is a collection of tools predicting outcomes of a group of interacting agents where an action of a single agent directly affects the payoff of other participating agents.

Types Of Games

A game can either be one in which moves (or choices) take place **sequentially** as in chess or one in which choices are made **simultaneously** such as a game of rock paper scissor. The distinction between simultaneous and sequential games is not so much about the timing of the moves, but rather about the information available to players when a move is made. In a sequential game, a player knows which particular choice her opponent has made from all those available to her, whereas simultaneous games involve players making choices prior to information becoming available about the choice made by the other. Business games are rarely, if ever, ones in which decisions are made exactly at the same point in time by all relevant firms. However, because it is often the case that companies must select from options before knowing what options rivals have selected, many business choices are best analysed as taking place within the framework of simultaneous games. Most actual games probably combine elements of both simultaneous and sequential move games.

Prisoner's Dilemma

The prisoner's dilemma is defined as follows: Two criminal accomplices are arrested and interrogated separately. Each suspect can either confess with a hope of a lighter sentence (defect) or refuse to talk (cooperate). The police does not have sufficient information to convict the suspects, unless at least one of them confesses. If they cooperate, then both will be convicted to minor offense and sentenced to a month in jail. If both defect, then both will be sentenced to jail for six months. If one confesses and the other does not, then the confessor will be released immediately but the other will be sentenced to nine months in jail. The police explains these outcomes to both suspects and tells each one that the other suspect knows the deal as well. Each suspect must choose his action without knowing what the other will do. A close look at the outcomes of different choices available to the suspects reveals that regardless of what one suspect chooses, the other suspect is better off by choosing to defect. Hence, both suspects choose to defect and stay in jail for six months, opting for a clearly less desirable outcome than only a month in jail, which would be the case if both chose to cooperate.

Nash Equilibrium

A set of strategic options is a Nash equilibrium if each player is doing the best possible given what the other is doing. Put another way, neither player would benefit by deviating unilaterally from the outcome, and so would not unilaterally alter its strategy given the opportunity to do so. In the above problem of prisoner's dilemma, Nash Equilibrium occurs if both of the prisoners defect. It is important to note that all games do not attain Nash Equilibrium.

Business And Game Theory

In any business, interactions with customers, suppliers, other business partners, and competitors play an integral role in any decision. Given that each firm is part of a complex web of interactions, any business decision or action taken by a firm impacts multiple entities that interact with or within that firm, and vice versa. Ignoring these interactions could lead to unexpected and potentially very undesirable outcomes. Game theory, is a very useful tool for studying interactive decision-making, where the outcome for each participant or "player" depends on the actions of others.

Game Trees

What makes game theory different from other analytical tools such as decision trees or optimization is that most analytical tools either take other parties actions as given, or try to model or predict them. Hence, their analysis focuses on optimizing from the perspective of one player and does not regard the strategic behaviour of other players.

EXAMPLE 1: Entry and Exit decisions

The manager of a firm is considering **the possibility of entering a new market**, where there is only one other firm operating. The managers decision will be based on:

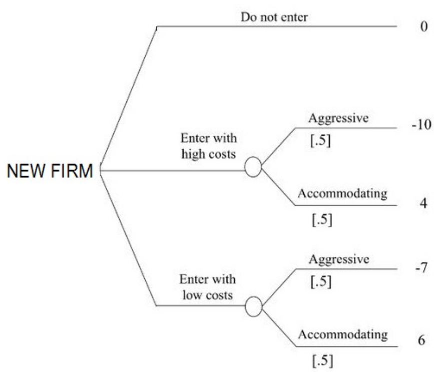
- **Profitability of market**, which in turn heavily depends on how the present firm will react to the entry. The present firm could be accommodating and let the entrant grab his share of the market or he could respond aggressively, meeting the entrant with a cut-throat price war.

- **Investment level of the entering firm**. The manager of the firm may invest to the latest technology and lower his operating costs (low cost case) or he may go ahead with the existing technology and have higher operating costs (high cost case).

The manager estimates that if his firm enters the market and the incumbent reacts aggressively, the total losses will be \$7 million in low cost case and \$10 million in high cost case. If the incumbent accommodates, however, the firm will enjoy a profit of \$6 million in low cost case and \$4 million in high cost case.

SOLVING THE PROBLEM

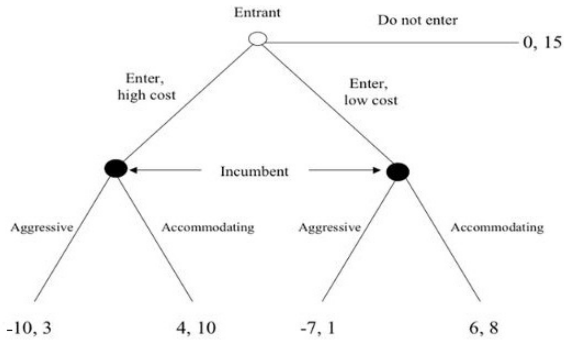
One possible approach for studying this problem is **decision analysis** which requires us to assess the probabilities for the incumbent being aggressive and accommodating. Assume that in this case, the manager thinks there is an equal chance of facing an aggressive and an accommodating rival. Given the estimated probabilities, we can draw the decision tree:



When we look at the profits, it is easy to see that if the manager chooses to enter he should invest in the latest technology. But still with a simple analysis, we see that it does not make sense to enter the new market, as in expectation, the company loses \$0.5 million. Can we conclude that the firm should not enter this market? What if the probabilities were not equal, and the probability of finding an accommodating rival was 0.55? **The point is, the manager's decision is very much dependent on the probabilities that he assessed for the incumbent's behaviour.**

As an **alternative approach**, we can use **game theory**. The best outcome for the incumbent is when she is the only one in the market. In this case, she would make a profit of say, \$15 million. If she chooses to be accommodating, her profits would be \$10 million if the entrant enters with the existing technology, i.e., high cost case, and \$8 million if he enters with the latest technology, i.e., low cost case. If she chooses to be aggressive, her profits would be \$3 million and \$1 million, respectively. Using the new information, we can draw a new tree, a game tree.

We can solve this problem by folding the tree backwards. If the firm were to enter, the best strategy for the incumbent is to accommodate. Knowing that this would be the case,



Game tree for the entry-exit model.

entering this new market would be worthwhile for the entrant. The same concept is used in another business strategy.

EXAMPLE 2: Price War

The players in the game we examine here are two firms, TopValue and PriceRite. Each firm must make a choice about the price at which it sells its product. To keep things as simple as possible, we suppose each firm has a simple binary choice: it can either cut its price (Cut) or leave it unchanged (Stick). The game is played just once.

PRICERITE →

TOP VALUE ↓

	CUT	STICK
CUT	2,2	4,1
STICK	1,4	3,3

As each of the two firms has two strategies available to it, there are four possible configurations of strategy, shown by the intersections of the row and column strategy choices in the matrix. The pair of numbers in each cell of the matrix denotes the 'payoff' (profit in this case) that each firm receives for a particular choice of option by Topvalue and PriceRite. The first number denotes the payoff to Topvalue, the second the payoff to PriceRite.

SOLVING:

To predict the outcome of this game, it is necessary to consider how the firms handle their strategic interdependence:

- NON CO-OPERATIVE SOLUTION: The first approach is to assume that each firm maximises its own profit, conditional on some expectation about how the other will act, and without collaboration taking place between the firms. One important concept that is widely used in looking for solutions to non-cooperative games is the idea of dominant strategy.

DOMINANT STRATEGY: A player has a dominant strategy when it has one strategy that offers a higher payoff than any other irrespective of the choice made by the other player. A widely accepted tenet of non-cooperative game theory is that dominant strategies are selected where they exist. Let us examine the payoff matrix to see whether either firm has a **dominant strategy**. First, look at the game from TopValue's point of view. If PriceRite chooses Cut, TopValue's preferred choice is Cut, as the payoff of 2 from cutting her price exceeds the payoff of 1 from sticking. Conversely, if PriceRite chooses Stick, TopValue's preferred option is Cut. We see that whatever PriceRite chooses, Cut is best for TopValue, and so is TopValue's dominant strategy. Similarly the dominant strategy for PriceRite is also Cut. Game theory analysis leads us to the conclusion that the equilibrium solution to this game consists of both firms cutting price.

Note that the outcome is inefficient. Both firms could do better if they had chosen Stick (in which case the profit to each would be three rather than two).

Why has this state of affairs come about?

There are two facets to the answer. The first is that the game has been played non-cooperatively. The second concerns the payoffs. These payoffs determine the structure of incentives facing the firms. In this case, the incentives are not conducive to the choice of Stick.

- CO-OPERATIVE SOLUTION AND ITS SUSTAINBILITY: Suppose that firms were to cooperate, making their choices jointly rather than separately. If both firms agreed to Stick and did what they agreed to do - payoffs to each would be 3 rather than 2 .

  But there is a problem: can these greater rewards be sustained? If self-interest governs behaviour, they probably cannot. To see why, note that the (Stick, Stick) outcome is not a Nash equilibrium. Each firm has an incentive to defect from the agreement to unilaterally alter its strategy once the agreement has been reached. Imagine that the two firms had agreed to Stick, and then look at the incentives facing TopValue. Given that PriceRite has declared that it will not cut its price, TopValue can obtain an advantage by defecting from the agreement (free-riding), leaving PriceRite to Stick - as agreed - but cutting price itself. In this way, firm TopValue could obtain a profit of 4. Exactly the same argument applies to PriceRite, of course. There is a strong incentive operating on each player to attempt to obtain the benefits of free-riding on the other's pollution abatement. These incentives to defect from the agreement mean that the cooperative solution is, at best, an unstable solution.

A POSSIBLE SOLUTION: A question that arises is, is it possible to transform this game in some way so that the (Stick, Stick) strategy pair becomes a stable cooperative solution?

There are ways in which this might be done. One possibility would be to negotiate an agreement with built-in penalty clauses for defection. For example, the agreement might specify that if either party defects (cuts) it must pay a fine of 3 to the other. If we construct the payoff matrix that would correspond to this agreement, it will be seen that the game structure has been transformed so that it is no longer a Prisoner's Dilemma game. Moreover, both firms would choose to abate.

EXAMPLE 3: Facebook buying Whatsapp

Social networking giant Facebook announced its decision to buy WhatsApp, a peer-to-peer messaging service linked through mobile numbers, for $19 billion (about Rs. 1,18,000 crore). A closer look at the fine nuances of how the bold move was probably guided reveals a carefully crafted strategy based on principles of game theory:
Facebook's strategy was simple: **price out rivals**. There are reports that Internet giant Google was also in active discussions to buy out WhatsApp. Initial reports had claimed that Google had bid $10 billion for the company, just about half of the amount Facebook offered. Later reports suggest that Google upped its offer for WhatsApp after it learned how much Facebook would eventually pay the messaging service. It is an example of a game theoretic exercise, where firms try to outthink each other.
It is best explained in the context of auctions. An auction is a non-cooperative game involving several players. Each player assumes that the other players would react in a certain way for every action of his. The strategies and tactics are based on this assumption. In mathematics, these are called as "mixed probabilities". Facebook's final price offer was based on a probability it had assigned about what other potential bidders- such as Google-may be willing to offer for WhatsApp. Likewise, Google's reported $10 billion offer was also based on what it thought was probable for a competitor to offer.

Conclusion

Finally, we recognise that circumstances will exist where a company will use a mix of competitive and cooperative behaviour. This will take us into the area of so-called 'competition'. It is important to note that the use of game theory along with other tools and the manager's business experience could significantly improve his understanding of the dynamics in business interactions and lead to higher quality and more informed decisions.

References

[1] Game Theory in Business Applications, paper by Feryal Erhun and Pinar Keskinocak, Stanford University
[2] www.wikipedia.org
[3] Game Theory, Avinash Dixit and Barry Nalebuff

NUPUR SOOD, B.Sc.(H) MATHEMATICS, 6th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN
E-mail address: nprsood@hotmail.com

LINEAR TIME INVARIANT THEORY

ADITI JAIN AND MANSI VERMA

ABSTRACT. The purpose of this document is to introduce basic concepts of LTI System Theory, the representation and analysis of LTI systems through convolution, their frequency response and application of LTI theory in acculumators and filters.

INTRODUCTION

What is a System? A system is a device that accepts an input signal x(t), processes it and gives an output signal y(t). The concept of system is applicable in the fields of electrical, economics etc. Physical devices such as filters, boilers, turbines are examples of systems. For example, filter rejects unwanted frequencies in the input signal. Similarly, for a car, the pressure on the accelator pedal is input signal and speed of the car is output signal.

$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$

Why are Systems used? Systems are used to perform signal processing. The effect of a system on the spectrum of signal can be analyzed easily if the system is LTI. LTI system is a system in which both linearity and time invariance holds. One of the primary reasons LTI systems are amenable to analysis is that if we can represent the input to an LTI system in terms of a linear combination of a set of basic signals, we can use superposition to compute the output of the system in terms of its responses to these basic signals. LTI theory comes from applied mathematics and has direct applications in technical areas such as seismology, NMR spectroscopy etc. It investigates the response of a linear and time-invariant system to an arbitrary input signal.

$x_1(t) + x_2(t) \longrightarrow \boxed{\text{Linear}} \longrightarrow y_1(t) + y_2(t)$

$x(t-\tau) \longrightarrow \boxed{\text{Time invariant}} \longrightarrow y(t-\tau)$

LTI systems are of two types :-

(1) Continuous time LTI system
(2) Discrete time LTI system

They are described by differential equation and difference equation respectively.

IMPULSE RESPONSE AND CONVOLUTION

Impulse response h(t) or h[n] of an LTI is the response to an impulse. i.e., $\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$. The significance of h(t) is that we can compute response to any input once we know response to impulse. The output of the system is the convolution of the input to the system with the help of system's impulse response

i.e. $\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$ (definition of Impuse Response)

$\delta(t-\tau) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t-\tau)$ (Time Invariance)

$x(\tau)\delta(t-\tau) \longrightarrow \boxed{\text{LTI}} \longrightarrow x(\tau)h(t-\tau)$ (Scaling Property)

$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\,d\tau \longrightarrow \boxed{\text{LTI}} \longrightarrow \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t)$ (Superposition Principle)

$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$ is known as convolution integral in case of Continuous Time and

$y[n] = \sum_{i=-\infty}^{+\infty} x[i]h[n-i]$ is known as convolution sum in case of Discrete Time LTI systems.

Frequency Response of LTI System

Let $x(t) = e^{\omega jt}$ be input into an LTI system with impulse response h(t. The output is

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= h(t) * e^{\omega jt} \\ &= \int_0^\infty h(\tau)e^{\omega j(t-\tau)}\,d\tau \\ &= e^{\omega jt}\int_0^\infty h(\tau)e^{\omega j(-\tau)}\,d\tau \\ &= H(j\omega)e^{\omega jt} \end{aligned}$$

where $H(j\omega) = \int_0^{+\infty} h[\tau]e^{-\omega j\tau}$ is known as Frequency Response.

Fourier Series Response of LTI System. Using LTI we can compute response to any linear combination of complex exponentials or sinusoids. If a complex sinusoid were input into an LTI system, then the output would be a complex sinusoid of the same frequency that has been scaled by the frequency response of the LTI system at that frequency.

The Fourier Series of a periodic signal is

$$C_n = \frac{\int_{-\infty}^{+\infty} x(t)e^{-\omega jnt}\,dt}{T} \text{ and}$$
$$x(t) = \sum_{i=-\infty}^{+\infty} C_n e^{\omega jnt}$$

Fourier Series Transform. Transform is used to convert signal from Time Domain to Frequency Domain. Fourier Transform of the above signal is given by

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-\omega jt}\,dt$$

For Discrete Time it is given as

$$X[e^{j\omega}] = \sum_{n=-\infty}^{+\infty} x[n]e^{-\omega jn}$$

As a result of the properties of these transforms, the output of the system in the frequency domain is the product of the transfer function and transform of the input. Then the output is obtained as a function of time by taking inverse Fourier Transform.

Laplace Transform. Continuous Time Fourier Transform provides us with representation of signals as linear combination of complex exponential of form e^{st} with $s = j\omega$. However, signals are not purely imaginary and can have real values as well. This leads to generalisation of Fourier transform known as Laplace Transform. i.e. $X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$

z-Transform. For a Discrete time linear invariant system with impulse response $h[n]$, the response $y[n]$ of the system to a complex exponential input of the form z^n is

$$y[n] = H(z)z^n, \text{ where } H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

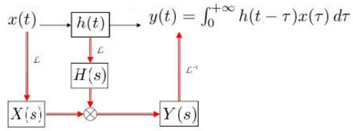


Figure 1

For $z = e^{j\omega}$ with ω real (i.e. $|z| = 1$), the summation in the above equation corresponds to the discrete-time Fourier transform of $h[n]$. More generally, when $|z|$ is not restricted to the unity, the summation is referred to as the z-transform of $h[n]$. The z-transform of a general discrete time signal $x[n]$ is defined as $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$, where z is a complex variable.

Properties of LTI System

Causality. A system is causal if the output at any time depends on values of the input at only the present and past times. For example, motion of automobile is causal since it does not anticipate action of the driver. For LTI system to be causal, $y(t)$ must not depend upon $x(\tau)$ for τ>t. For this to be true all coefficients of $h(t - \tau)$ that multiply values of $x(\tau)$ for τ>t must be zero. This then requires that impulse response of causal LTI system satisfy the condition $h(t) = 0$ for t<0.

Stability. A stable system is one in which small input signals lead to responses that do not diverge. If an input signal is bounded then the output signal must be bounded.

$$\forall x\colon |x| < U \longrightarrow |y| < V.$$

For LTI system to be stable, its impulse response must be absolutely summable i.e. $\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$. Looking at unit impulse response, allows us to determine certain system properties.
For example, Figure 2 shows the causal, stable, finite impulse response
$y[n] = x[n] + 0.5x[n-1] + 0.25x[n-2]$

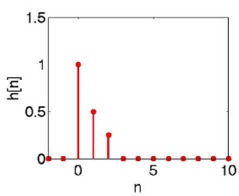


Figure 2

Applications of LTI Theory

Accumulator. Consider a DT LTI system with an impulse response

$$h[n] = u[n]$$

Using convolution the response to an arbitrary input $x[n]$ is

$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$

As $u[n-k] = 0$ for n-k<0 and 1 for n-k>0, this becomes

$y[n] = \sum_{k=-\infty}^{+\infty} x[k]$

i.e., it acts as a running sum or accumulator.

Inverse Filtering. Let's assume that some naturally occuring process introduces unwanted filtering to a signal $s(t)$ and we are asked to design an inverse filter to remove the unwanted filtering. The input to our inverse filtering system is the distorted signal $d(t) = h_d(t) * s(t)$ where h_d is the impulse response function of the distorting system. We can express the periodically extended input signal via its Fourier Series

$$d_e(t) = \sum_{m=-\infty}^{+\infty} H_d(\tfrac{2\pi jm}{T_o}) S_m e^{\frac{j2\pi mt}{T_o}},$$

where $H_d(\frac{2\pi jm}{T_o})$ is the laplace transform of impulse response of distorting system filter and S_m the Fourier coefficients of the undistorted signal $s(t)$. The output from our inverse filtering system is then given by the Fourier series

$$y_e(t) = \sum_{m=-\infty}^{+\infty} \hat{H}(\tfrac{2\pi jm}{T_o}) H_d(\tfrac{2\pi jm}{T_o}) S_m e^{\frac{j2\pi mt}{T_o}}$$

where $\hat{H}(j\omega_m)$ is our Laplace transform of impulse response of inverse filter that we wish to design. Since we want the output $y_e(t)$ to be equal to $s_e(t)$ we then require that

$$\hat{H}(\tfrac{2\pi jm}{T_o}) H_d(\tfrac{2\pi jm}{T_o}) = 1 \Longrightarrow \hat{H}(\tfrac{2\pi jm}{T_o}) = \frac{1}{H_d(\frac{2\pi jm}{T_o})}$$

It appears from the above equation that we have solved the problem, i.e., have designed the necessary inverse filter. The problem is that we have only so much freedom in physically building analog filters. These filters have to be constructed using physical components that are subject to the laws of physics and have serious constraints placed upon them because of this. For example, analog filters $\hat{H}(j\omega_m)$ described by linear ODEs is in the form of a ratio of polynomials

$$\hat{H}(j\omega_m) = D_x(j\omega_m)H(j\omega_m) = \frac{D_x(j\omega_m)}{D(j\omega_m)}$$

where $D_x(j\omega_m)$ is the characteristic polynomial of the differential operator relating the standard and physical inputs to the system and where $D(j\omega_m)$ is the characteristic polynomial of the system ODE. On the other hand, there is no guarantee that the distorting filter $H_d(\frac{2\pi jm}{T_o})$ has such a mathematical form so that no exact solution may be possible. An approximate solution is, of course, possible where we approximate the H_d^{-1} by the use of polynomials.

References

[1] Professor E. Yagle, Andrew, Dept. of EECS, The University of Michigan, An Introduction to Linear Time-Invariant Systems and Their Frequency Response

[2] Devaney, A.J., Department of Electrical and Computer Engineering, Northeastern University Boston, December 29, 2005. An Introduction to Linear Systems Theory

[3] Oppenheim, A.V. Signals and Systems.New Delhi: PHI private limited, 1997.

ADITI JAIN, MANSI VERMA, B.Sc.(H) MATHEMATICS, 4th SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN

E-mail address: aditi1990jain@yahoo.com, mansivermalsr@gmail.com

MATHEMATICS IN SOLVING CRIMES.

ANKITA TULSHYAN

Abstract. Small mathematics is typically perceived as a tough but important subject. While its theoretical merit is easily appreciated, it is quite unfortunate that its vast practical applications are not discussed enough. One exciting discipline where use of mathematical skills goes unrecognized is in solving crimes. Imagine making a formula for estimating the geographical location of a serial killer, or using probability theory to solve a forgery case, making and cracking secret codes. This article attempts to provide an introduction to a few such techniques which are successfully used in solving criminal cases.

Introduction

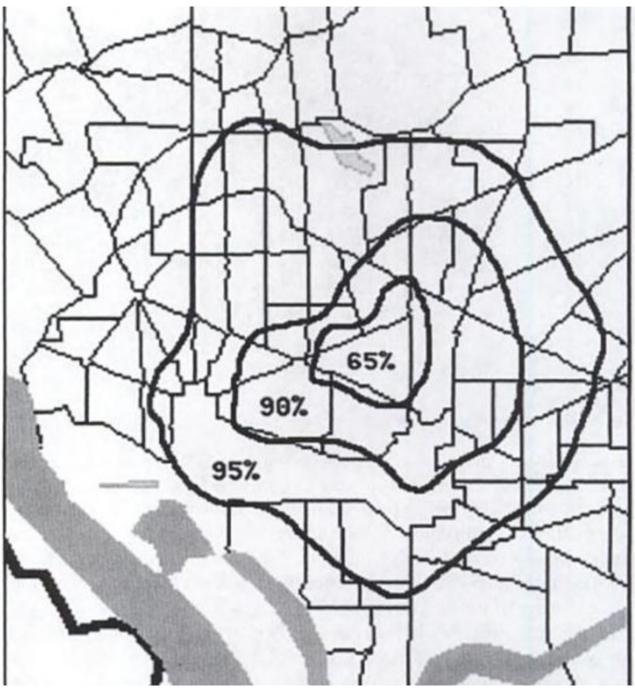
Who has not read Sir Doyle's 'The Adventure of The Dancing Men'? Today famous television series like NUMB3RS and SHERLOCK have made mathematics a seemingly more compatible discipline with criminology, but still people fear and see mathematics as a distant relative of criminology. Well, mathematics itself seems like a product of crime for many people. With technology evolving rapidly in last few decades, crime rate has gone up drastically as criminals are coming up with new ideas. As a result, we need to embrace new methods to fight these ever increasing crimes. DNA profiling, Image Analysis, Probability Theory, Game Theory, Statistics, Prime Number Theory, these are some among the many sections which include mathematics and are used by various agencies to fight criminals. We will be discussing briefly about some of the techniques used by various forces to catch criminals.

Geographic Profiling

Imagine that 10 people are killed in your city, in the last 4 months, random people with no connection, killed at random places, just the way of killing is same, which gives the idea of murderer being a serial killer. At first it seems impossible to catch such a criminal, there being no connection or pattern between different killings, but mathematics says a pattern has to be there. This is the idea which was further worked upon and as a result a formula for finding such a criminal was formed. Before undertanding how mathematics helps in knowing the estimated location of such a criminal's residence, let's first understand, Geographic profiling. It is a method that analyzes the locations of a connected series of crimes to determine the most probable area of the criminal's residence. Mostly, it is used in cases of serial murder or rape.

The leading developer of geographic profiling is Dr. Kim Rossomo. He is a former police officer, who went on to take Ph.D in criminology. His thesis advisers, Paul and Patricia

Brantingham, were pioneers in the development of mathematical models of criminal behavior, particularly those that describe where crimes are most likely to occur based on where a criminal lives, works, and plays. He did not want to study criminal behaviour, but wanted to use actual data about the locations of crimes linked to a single unknown criminal as an investigative tool. Working on it for a long time, he finally hit upon the idea and came up with the formula which is discussed below.
At first, the formula seems to be a complex one , but it is easy to understand. Consider a map divided into squares marked as (x,y), x denoting row and y denoting columns. The formula takes input from the past crime locations and gives the probability of the position of the serial criminal residing within a specific area by summation of the locations where the criminal has committed the past crimes. The figure below is an example of how an area generated by such a formula look like.



$$p_{i,j} = k \sum_{n=1}^{totalcrimes} [\frac{\phi_{i,j}}{(|X_i - x_n| + |Y_j - y_n|)^f} + \frac{1 - \phi_{i,j}(B^{g-f})}{(2B - (|X_i - x_n| + |Y_j - y_n|)^g}] \quad (1)$$

Where, B is the buffer zone area (the neighbourhood of a criminal residence), and

$$\phi ij = \begin{cases} 1, & \text{if } (|X_i - x_n| + |Y_j - y_n|) > B \\ 0, & else \end{cases} \quad (2)$$

ϕij is a characteristic function that returns 0 when a point is an element of the buffer zone B, $|X_i - x_n| + |Y_j - y_n|$ is the Manhattan distance between a point and the n^{th} crime site.
The constant k is empirically determined. Also, the variables f and g are chosen so that it works best on the data of past crimes.

The main idea on which this formula works is that the probability of crimes first increases as one move through the buffer zone away from the hot zone, but decreases afterwards.

With later refinements, the formula has become the principal element of a computer program Rossmo wrote, called Rigel. Today, Rossmo sells Rigel, along with training and consultancy, to police and other investigative agencies around the world to help them find criminals. The formula has been applied to fields other than forensics also.

Estimating the time of death- Differential equation

A person is found dead in his locked apartment in South- Delhi early in the morning at 7:00 A.M., nobody knows the time of his death. We might say that it seems like there is no way to find out his time of death and cause. However it is not true, the answer can be found very easily. Here, mathematics and physics come to our aid to find an approximate time of the person's death. Using Newtons law of cooling, it is possible to form a differential equation model to find the timing of the death of the person. So, lets first briefly understand the Newtons law of cooling.
It states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. Thus, the temperature of the body $T(t)$ governed by Newton's Law of Cooling satisfies the following differential equation,

$$\frac{dT}{dt} = k[T(t) - T_a(t)] \tag{3}$$

Where k is a negative constant, T_a(t) is the ambient temperature, and in our case t is the number of hours since the death of the person.
We already know the body was discovered at 7:00 A.M. The coroner measured its temperature, T_1 at that time. One hour later another temperature, T_2 was taken. The temperature of the victim's apartment was found to be constant at a temperature of 70°F. The given differential equation can be solved to get,

$$T(t) = T_a + (T_0 - T_a)e^{-kt}, \tag{4}$$

where T_0 is the normal temperature of body, i.e, 98.6°F. Thus the equation in our case reduces to-

$$T(t) = 70 + (28.6)e^{-kt} \tag{5}$$

Let t_c be the time when coroner first checks the temperature of the body, thus we have the two equations,

$$72.5 = 70 + (28.6)e^{-kt_c} \tag{6}$$

and

$$72 = 70 + (28.6)e^{(-kt_c+1)} \tag{7}$$

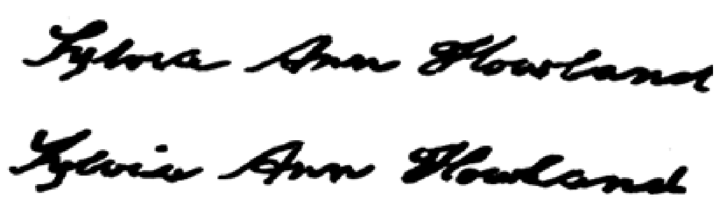
Solving the two equations we get value,

k = - (0.233)
and, t_c = 10.92 hours

From this calculated value we can say that the man died about 11 hours before 7:00 A.M., which would be around 8:00 P.M., the previous evening. Thus, the application of mathematics and physics together, gives us the estimated time of the victim's murder.

Mathematics in Courtroom

Mathematics is not only used in finding criminals but also in courtrooms. It can be used to check the reliability of DNA results, recognizing the criminal using the technique of Image Enhancement, detecting a forgery case with the help of probability theorems, and many more. To illustrate this, let's discuss a forgery case which was solved with the aid of mathematics.
In the 19th century, Benjamin Peirce was one of the leading mathematics professor at Harvard University, and his son, Charles Sanders Peirce, was also a brilliant scholar. It was a forgery trial involving the estate of Sylvia Ann Howland. The case was to check the authenticity of her signature on the will produced by her niece Hetty, which made Hetty the sole benefactor of Sylvia's wealth. In most forgery cases, someone attempts to duplicate the signature, so by showing the dissimilarities between the two signatures the case can be solved but in this case the forgery was simply too good!
Here are the two signatures, They devised a method to compare and express the agreement



between any two signatures as a score. To determine this score, they decided to use downstrokes; there are thirty of them in each signature.
They obtained a set of forty-two authentic signatures of Sylvia. For forty-two signatures there are

$$\frac{(42\times41)}{2} \equiv 861$$

ways to select a pair of signatures to compare. For each of these 861 pairs, they determined the number of downstrokes that coincided. They found a total of 5,325 coincidences among the 25,830 (861 x 30) comparisons of downstrokes. That meant that about one out of five comparisons was judged a perfect match, occurring with probability,

$$\frac{5325}{25830} = 0.206156$$

The rest of their analysis was mathematical, or more specifically, statistical. Assuming these coincidences occur independently, Peirce used the product rule to find the chance of all thirty downstokes being identical, i.e.

$$(.206156 \times .206156 \times .206156 \times ...)[30 times]$$

This is approximately 1 in 375 trillion.

Professor Peirce summarized his findings in this way: *"So vast improbability is practically an impossibility. Such evanescent shadows of probability cannot belong to actual life. They are unimaginably less than those least things which the law cares not for. ... The coincidence which has occurred here must have had its origin in an intention to produce it. It is utterly repugnant to sound reason to attribute this coincidence to any cause but design."*
Surely not surprising, the court ruled against Hetty Robinson.
A modern mathematician may solve such a case today with different statistics with more firm ground and less assumptions like the independency of the occurrence of two co-incidental downstokes, but still Peirces method opened a new way for future mathematicians.

Conclusion

Thus, we learnt about some of the interesting techniques used for catching a criminal, which use mathematics. Mathematics has many more such wonderful applications.

References

[1] Devlin, K.J.,& Lorden, G. (2007) *The numbers behind NUMBERS, Solving crime with mathematics.* New York: Plume

[2] *Can math and science help solve crimes, UCLA Newsroom.*, http://newsroom.ucla.edu/portal/ucla/can-math-and-science-help-solve-153986.aspx

[3] *Estimating Time of Death.* http://people.uncw.edu/lugo/MCP/DIFF_EQ/deproj/death/death.htm

[4] *Rossmo's Formula- Geographic Profiling.* https://sites.google.com/site/gs2013m3/rossmo-s-formula

ANKITA TULSHYAN, B.Sc.(H) MATHEMATICS, 2nd SEMESTER, LADY SHRI RAM COLLEGE FOR WOMEN

E-mail address: ankitatulshyan54@gmail.com