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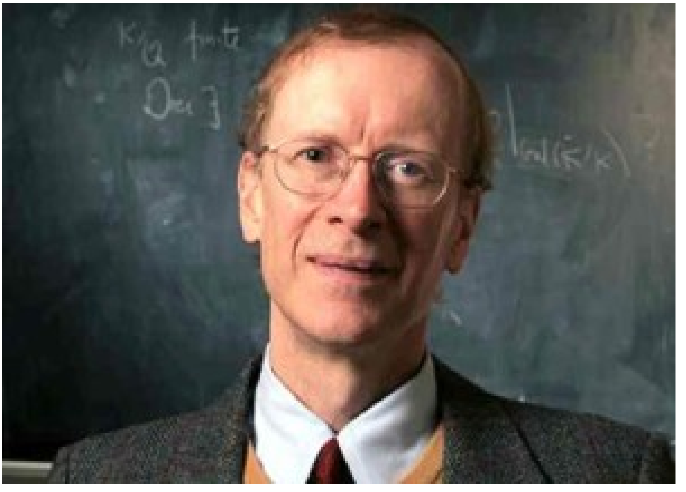

ANDREW WILES

This journal is dedicated to Sir Andrew J. Wiles, the proud recipient of the Abel Prize 2016 widely regarded as the Nobel for mathematics.

Sir Wiles who is currently a professor at Oxford is getting the prestigious honor for his stunning proof of Fermat's Last Theorem, published in 1995, by way of the modularity conjecture for semistable elliptic curves, opening a new era in number theory.

Fermat's Last Theorem was formulated by Pierre de Fermat in 1637, it states:

"There are no whole number solutions to the equation $x^n + y^n = z^n$ for n $>$ 2."

PREFACE

This edition Éclat, 2015 - 2016 has been an extraordinary expedition taking us through ten intense mathematics papers with incessant rounds of editing to bring out the final version. The journal, which was only an idea a few years back has become an integral part of the Mathematics department of the college. We hope that the readers will get a detailed knowledge of the various topics of mathematics. It has been a great challenge as well as a pleasure to put together research and insights into black and white.

This edition of Éclat contains four categories of mathematics namely, History of mathematics, Rigour in mathematics, Inter disciplinary aspects of mathematics and Extension of course content. We have tried our best to make the journal inclusive so that readers from all fields can find a suitable content of their interest. This journal along with increasing your mathematical understanding will also stimulate your thought process to come out with new ideas.

The publication of this journal would have been impossible without the enthusiasm of the department of Mathematics of our college. We sincerely thank the faculty of the Department of Mathematics, Lady Shri Ram College for Women, for guiding and supporting us throughout the year. We are open to any suggestions, corrections and submission from our readers.

We dedicate this volume of the journal to Andrew Wiles, recipient of Abel Prize 2016.

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CONTENTS

History of Mathematics

Mathematics is the one of the oldest academic discipline involving stimulating and intriguing concepts. It is far beyond the ken of one individual and to make any contribution to the evolution of ideas, an understanding of the motivation behind the ideas is needed. The section covers the genesis of mathematical ideas, the stream of thought that created the problem and what led to its solution. The aim is to acquaint the readers with historically important mathematical vignettes and make them inured in some important ideas of Mathematics.

BOURBAKI: A SECRET SOCIETY OF MATHEMATICS

SHRUTI KAUSHAL

Abstract

The Bourbaki group, officially known as the *Association des collaborateurs de Nicolas Bourbaki* (Association of Collaborators of Nicolas Bourbaki), was a collective using the pseudonym Nicolas Bourbaki under which a group of (mainly French) 20th-century mathematicians wrote a series of books.The Bourbaki writings commenced in 1939 with the first volume of their *Elements de mathematique* ("Elements of Mathematics"). Their work led to the discovery of several concepts and terminologies still used, and influenced modern branches of mathematics. This paper retraces Bourbaki's initial goals and analyzes some of the first written outlines for the treatise. It shows how Bourbaki set out to write a series of text on analysis with the aim of revolutionizing the teaching methodology prevalent in the early 1900's.

1 Introduction and History

Bourbaki was born in Paris in 1935 when a small group of mathematicians at the *Ecole Normale Suprieure*, dissatisfied with the courses they were teaching, decided to reformulate them.The would-be members of Bourbaki met for the first time to discuss the project at the end of 1934 in a Parisian cafe.
Andre Weil stated their collective goal, quite explicitly, as -

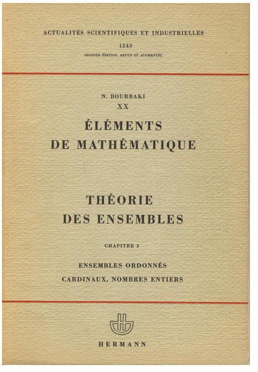
> "to define for twenty-five years the syllabus for the certificate in differential and integral calculus by writing, collectively, a treatise on analysis. Of course, this treatise will be as modern as possible" (Beaulieu, 1993, p. 28)

The founding members of the group included *Henri Cartan*, *Claude Chevalley*, *Jean Coulomb, Jean Delsarte, Jean Dieudonn, Charles Ehresmann, Szolem Mandelbrojt, Rene de Possel*, and *Andre Weil*. *Cartan, Chevalley, Delsarte, Dieudonne*, and *Weil*, all former students of the *Ecole Normale SupErieure*, remained the most influential and active force within the group for decades.

Bourbaki Congress, 1938

There was no official status of membership. Moreover, at that time the group was quite secretive and also fond of supplying disinformation. Regular meetings were scheduled (about 4 weeks a year), during which the group would discuss vigorously every proposed line of every book. At each meeting, individual members were commissioned to produce drafts of the different chapters. The drafts were subjected to harsh criticism by the other members and then were reassigned for revision. Only after several drafts had been written and criticized was the final document ready for publication. Every chapter and every volume of Bourbaki's treatise was the outcome of arduous collective work, and the spirit and point of view of the people who had written it were hardly recognizable.
One of the members, *Dieudonne*, wrote that anyone attending a meeting for the first time would come out with the impression that it is a gathering of madmen. He could not imagine how these people, shouting - sometimes three or four at the same time, could ever come up with something intelligent. Bourbaki's own rules explicitly provided for self-renewal: from time to time, younger mathematicians were invited to join and older members resigned, in accordance with mandatory "retirement" at age fifty.
Bourbaki was a fervent believer in the unity and universality of mathematics, and dedicated itself to demonstrating both by recasting all of mathematics into a unified whole. Its goals were total formalization and perfect rigor. In the post-war years, Bourbaki metamorphosed from rebel to establishment. This can well be validated by the title chosen for the group's work- *Elements de Mathematique*. Armand Borel (a member of the Bourbaki) explained the subtle title that was chosen as follows-

> The title "*Elements de Mathematique*" was chosen in 1938. It is worth noting that they chose "*Mathematique*" rather than the much more usual "*Mathematiques*". The absence of the "*s*", was of course, quite intentional, a way for Bourbaki to signal its belief in the unity of mathematics.



Elements de Mathematique, 1939

The first volume of the series of mathematical texts, *Elements de Mathematique* ("Elements of Mathematics"), by the group, was published in 1939. The series covers various theories namely: set theory, topology, topological vector spaces, algebra, commutative algebra, functions of one variable, integration etc.. It aimed to be a completely self-contained treatment of the core areas of modern mathematics. Assuming no special knowledge of mathematics, it tried to take up mathematics from the very beginning, proceed axiomatically and give complete proofs.
The still-incomplete series of more than 30 monographs soon became a standard reference on the fundamental aspects of modern mathematics and the various historical notes included at the end of the chapters were published as a collection in 1960 in *Elements d'histoire des mathematiques* ("History of the Elements of Mathematics").

The Bourbaki Group was able to stir things up in the world of mathematicians, but not all luminaries were impressed by its work. Since the group choose to operate under a pseudonym, it shrouded itself in mock mystery and this led to the creation of a **Bourbaki myth**. The group and its work stood in the mathematical lime light, especially in the 1950's and 1960's. Some mathematicians approved strongly of Bourbaki's projects and some condemned it completely. Emil Artin, an Austrian mathematician, was one of the many eminent personalities who supported the group's work.

> "Our time is witnessing the creation of a monumental work: an exposition of the whole of present day mathematics. Moreover this exposition is done in such a way that the common bond between the

> various branches of mathematics become clearly visible, that the framework which supports the whole structure is not apt to become obsolete in a very short time, and that it can easily absorb new ideas."

-Emil Artin

The group's work provoked some hostility too but mostly on the side of classical analysts. The presentation of the texts/work was the main area of concern. Many mathematicians believed that because of the high abstraction of topics like geometry and topology, it was almost impossible for the reader to truly understand the topic. The books were austere, dogmatic and pedagogical. Pierre Cartier, member of the group for the period 1955-1983, was quoted as saying-

> "essentially no analysis beyond the foundations: nothing about partial differential equations, nothing about probability. There is also nothing about combinatorics, nothing about algebraic topology, nothing about concrete geometry. And Bourbaki never seriously considered logic. Anything connected with mathematical physics is totally absent from Bourbaki's text. The Bourbaki were *Puritans* and *Puritans* are strongly opposed to representations of truths of their faith."

Furthermore, the group was criticized for reducing geometry as a whole to abstract algebra and soft analysis. Less emphasis on problem solving than axiomatic theory building and no discussion on Combinatorics are some of the many issues because of which many mathematicians deemed the group's work uninteresting.

2 Contribution to Mathematics

2.1 Symbols and Terms

The two most popular mathematical symbols given by the Bourbaki group include the empty set notation - $\emptyset$ and the 'dangerous bend symbol'.
The terms - 'injective', 'surjective' and 'bijective' were coined by the Bourbaki group.

2.1.1 Empty set ($\emptyset$)

The symbol $\emptyset$ was introduced by the Bourbaki group (specially *Andre Weil*) in 1939 after being inspired by the letter $\varnothing$ in Norwegian and Danish. It has the sound 'er'. It appeared in the first text by the group on set theory.
Please note that the symbol for the empty set $\emptyset$, the Norwegian letter, is completely and totally unrelated to the Greek letter ϕ (pronounced 'phi'). The letter ϕ is used to represent various functions and numbers in mathematics, like the golden ratio, Euler's toitent function, Ring homomorphisms etc., but not the null set.

2.1.2 Dangerous Bend Symbol

The 'dangerous bend symbol' was first used by the Bourbaki group in their books. It is used to mark tricky passages and resembles a road sign which indicates "dangerous bend". It appears in the margins of many books authored by the Bourbaki.

☡ ☡ ☡

The description of the symbol by Nicholas Bourbaki (pseudonym of the group),given in many Bourbaki textbooks, is as follows-

> Some passages are designed to forewarn the reader against serious errors, where he risks falling; these passages are signposted in the margin with the sign **Z** ("dangerous bend")

2.2 Theorems

2.2.1 Jacobson-Bourbaki Theorem

Statement- Suppose that **L** is a division ring. The Jacobson-Bourbaki Theorem states that there is a natural $1:1$ correspondence between-

1. Division rings K in **L** of finite index n. (In other words **L** is a finite dimensional left vector space over K).
2. Unital K algebras of finite dimension n (as K-vector spaces) contained in the ring of endomorphisms of the additive group of K.

The sub division ring and the corresponding subalgebra are each others commutants.

The proof of this theorem is beyond the scope of the article.

2.2.2 Bourbaki-Witt Theorem

The Bourbaki Witt Theorem in order theory, named after Nicholas Bourbaki and Ernst Witt, a German mathematician, is a basic fixed point theorem for partially ordered sets.

Statement- It states that if X is any non-empty chain complete poset,and

$$f : X \longrightarrow X$$

such that

$$f(x) \geq x, \forall x$$

then f has a fixed point. Such a function f is called inflationary or progressive.

The proof of this theorem is beyond the scope of this article

Bourbaki-Witt has many applications. For example, in computer science, it is used in the theory of computable functions. It is also used to define recursive data types, e.g. linked lists, in domain theory.

3 Conclusion

While several of Bourbaki's books have become standard references in their fields, some have felt that the austere presentation makes them unsuitable as textbooks. The books' influence may have been at its strongest when few other graduate-level texts in current pure mathematics were available, between 1950 and 1960.

In my opinion, the voluminous work done by the Bourbaki, with the aim of laying universal foundations, was in danger of sinking under its own weight. The text seems to be a mere collection of facts and axioms; more like an encyclopedia and encyclopedias are not textbooks. Much of the critique directed against Bourbaki is that it was used, or perhaps misused, to reform school education. Bourbaki was to some extent the victim of its own success. The group was led astray, from the path of bringing about an educational revolution, by the enthusiasm and the success of recruiting many of the leading mathematicians of the time. This broadened their horizons and all of the mathematics had to be included and hence the presentation (of the text) suffered.

Association des collaborateurs de Nicolas Bourbaki (Association of Collaborators of Nicolas Bourbaki), has an office at the cole Normale Suprieure in Paris.

Bourbaki has indeed shaken up the world of mathematics. Some people approve of its ways and some don't, but one cannot deny Bourbaki's influence.

It is now left to the reader to decide whether The Bourbaki group was successful in reforming the mathematical text available till present.

References

[1] www.ega-math.narod.ru/Bbaki/Bourb3.htm

[2] https://ncatlab.org/nlab/show/Bourbaki

[3] www.numericana.com/fame/bourbaki.htm

[4] https://www.sciencenews.org/article/founder-secret-society-mathematicians

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MONTE CARLO SIMULATIONS

ANIKA JAIN

Abstract

Is it possible to accurately predict the future of the actions we take? When asked such a question an immediate answer is no. But what if we can get close enough to the real scenario Risk analysis is part of every decision we make. Even though we have unprecedented access to information, we can't accurately predict the future. We are constantly faced with uncertainty, ambiguity, and variability. Monte Carlo simulation (also known as the Monte Carlo Method) lets us see all the possible outcomes of our decision and assess the impact of risk, allowing for better decision making under uncertainty. Monte Carlo simulation is a computerized mathematical technique that allows people to account for risk in quantitative analysis and decision making. The technique is used by professionals in numerous fields.

"At each stage of in the advance of mathematical thought
the outstanding characteristics are novelty and originality."
-George Frederick James Temple

The Monte Carlo methods are statistical sampling techniques that over the years have been applied successfully to a vast number of scientific problems. They are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results. They are often used in physical and mathematical problems and are most useful when it is difficult or impossible to use other mathematical methods. Monte Carlo methods are mainly used in three distinct problem classes: optimization, numerical integration, and generating draws from a probability distribution.

Monte Carlo methods vary, but tend to follow a particular pattern:

1. Define a domain of possible inputs.
2. Generate inputs randomly from a probability distribution over the domain.
3. Perform a deterministic computation on the inputs.
4. Aggregate the results.

In a Monte Carlo simulation, the entire system is simulated a large number of times. Each simulation is equally likely, referred to as a realization of the system. For each realization, all of the uncertain parameters are sampled (i.e., a single random value is selected from the specified distribution describing each parameter). The system is then simulated through time (given the particular set of input parameters) such that the performance of the system can be computed. This results is a large number of separate and independent results, each representing a possible "future" for the system (i.e., one possible path the system may follow through time). The results of the independent system realizations are assembled into probability distributions of possible outcomes. As a result, the outputs are not single values, but probability distributions.

1 The Beginning

During the wartime period, a team of scientists, engineers and technicians were working on the first electronic computer called the ENIAC at the University of Pennsylvania in Philadelphia. Their mentors were Physicist John Mauchly and Engineer Presper Eckert. Mauchly had realized that if electronic circuits could count, then they could do arithmetic and hence solve differential equations at incredible speeds. He proposed the idea to the Ballistics Research Laboratory at Aberdeen that an electronic computer be build to deal with such calculations.

John von Neumann

John von Neumann was a Hungarian-American pure and applied mathematician, physicist and polymath who has about 150 research papers to his credit. He was a pioneer of the application of operator theory to quantum mechanics, in development of functional analysis, a key figure in the development of game theory and the concepts of cellular automata, the universal constructor and the digital computer. At that time he was a Professor of Mathematics at the Institute for Advanced Study in Princeton and was also a consultant at Aberdeen and Los Alamos. He became seriously interested in the thermonuclear problem being pawned at that time in Los Alamos by fellow Hungarian scientist Edward Teller. He asked Stan Frenkel and N. Metropolis to prepare a preliminary computational model of the thermonuclear reaction for the ENIAC.

In March 1945, they visited the Moore School of Electrical Engineering at University of Pennsylvania for an advance glimpse of the ENIAC. It was a huge physical structure with 18,000 double triode vaccum tubes and 500,000 solder joints. After becoming saturated with the indoctrination about the general and physical structure of the ENIAC, Frankel and Metropolis returned to Los Alamos to work on a model that was realistically calculable.

After the completion of the thermonuclear computations a review of the ENIAC results was held in the spring of 1946 at Los Alamos. The principals included Edward Teller, Enrico Fermi, John von Neumann, and the Director, Norris Bradbury. Frankel, Metropolis and Anthony Turkevich described the calculations and the conclusions. The simplifications of the model were taken into account and the extrapolated results were the cause of guarded optimism and feasibility of a thermonuclear weapon.

Among the attendees was **Stanislaw Ulam**, an excellent Polish-American mathematician who had rejoined the Laboratory. He became aware of the electro-mechanical computers used for implosion studies and was duly impressed by the speed and versatility of the ENIAC. His extensive mathematical background made him aware that statistical sampling technique had fallen into desuetude because of the length and tediousness of the calculations. While playing solitaire during his recovery from surgery, Ulam had thought about playing hundreds of games to estimate statistically the probability of a successful outcome. With the miraculous development of the ENIAC, it occurred to him that statistical techniques should be resuscitated and he discussed this idea with von Neumann. Thus was triggered the spark that led to the Monte Carlo Method.

Stanislaw Ulam

2 The Letter

John von Neumann saw the relevance of Ulam's suggestion.
The approach of doing statistical sampling using computing techniques seemed to be suitable for exploring behavior of neutron chain reactions in fission devices. In particular, neutron multiplication rates could be estimated and used to predict the explosive behavior of the various fission weapons then being developed.

On 11 March 1947, von Neumann sent a handwritten letter to Robert Richtmyer, the Theoretical Division leader. His letter included a detailed outline of a possible statistical approach to solving the problem of neutron diffusion in fissionable material. This outline was the first formulation of a Monte Carlo computation for an electronic computing machine.

In his formulation von Neumann considered a spherical cone of fissionable material surrounded by a shell of tamper material. He assumed some initial distribution of the neutrons generated isotropically with known velocity spectrum. The simple model ignored radioactive and hydrodynamic effects. The idea was to follow the development of a large number of individual neutron chains as a consequence of scattering, absorption, fission and escape.
At each stage a sequence of decisions has to be made based on statistical probabilities appropriate to the physical and geometric factors. The first two decisions occur at time t=0, when a neutron is selected at random to have a certain velocity and a certain spatial position. The next decisions are the position of the first collision and the nature of that collision.

If it is determined that a fission occurs, the number of emerging neutrons must be decided upon, and each of these neutrons is eventually followed in the same fashion as the first. If the collision is decreed to be a scattering, appropriate statistics are invoked to determine the new momentum of the neutron. When the neutron crosses a material boundary, the parameters and characteristics of the new medium are taken into account.Thus, a genealogical history of an individual neutron is developed. The process is repeated for other neutrons until a statistically valid picture is generated.

At the end of the letter, von Neumann attached a tentative 'computing sheet' that he felt would serve as a basis for setting up the calculations on the ENIAC and assured that it would not exceed the logical capacity of the ENIAC. He further stated that in changing over problems, only a few numerical constants will have to be set anew on one of the 'function table' organs of the ENIAC. He briefly discussed the changes caused by hydrodynamics for an explosibe device and treatment of the radiation that is generated during fission.

He stated optimistically that *"the approach will gradually lead to a completely satisfactory theory of efficiency, and ultimately permit prediction of the behavior of all possible arrangements."*

In 1947, the method was quickly brought to bear on problems pertaining to thermonuclear and fission devices. In 1948, Stan was able to report to the Atomic Energy Commission about the applicability of the method for such things as the cosmic ray showers and the study of Hamilton Jacobi partial differential equations. All ensuing work on Monte Carlo neutron-transport codes for weapon development has been directed at implementing the details layed out in von Neumanns' 1947 letter.

3 Method

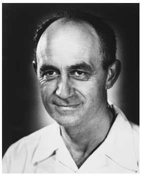
Being secret, the work of von Neumann and Ulam required a code name. N. Metropolis suggested the name Monte Carlo, which refers to the Monte Carlo Casino in Monaco where Ulam's uncle would borrow money from relatives to gamble.

In order to make various decisions, it was important for the computer to have a source of uniformly distributed pseudo-random numbers. John von Neumann developed an algorithm for generating such numbers called the 'middle-square method'. Here, an arbitrary n-digit number is squared, creating a $2n$-digit product. A new integer is formed by extracting the middle n-digits from the product. This forms a chain of numbers which repeats after some time. H. Lehmer suggested a scheme based on Kronecker-Weyl theorem that generated all possible numbers of n digit before it repeats.

After forming an algorithm for generating a uniformly distributed set of random numbers, these numbers must be transformed into the non uniform distribution 'g' that simulate probability distribution functions specific to each particular type of decision. Function needed to achieve this transformation is $f = g^{-1}$.

Monte Carlo methods were central to the simulations required for the Manhattan Project, though severely limited by the computational tools at the time. In the 1950s they were used at Los Alamos for early work relating to the development of the hydrogen bomb, and became popularized in the fields of physics, physical chemistry, and operations research. The Rand Corporation and the U.S. Air Force were two of the major organizations responsible for funding and disseminating information on Monte Carlo methods during this time, and they began to find a wide application in many different fields.

4 An Analog Computer: The FERMIAC



Enrico Fermi

Enrico Fermi was an Italian physicist often called the 'architect of the nuclear age' who excelled both theoretically and experimentally. In the early thirties, he was particularly interested in neutron diffusion. Nearly fifteen years earlier, he had independently invented the present Monte Carlo method while studying the moderation of neutrons in Rome. While he did not publish anything, he used the method to solve many problems and took great delight in astonishing his Roman colleagues with his remarkably accurate predictions of experimental results. During the hiatus in the ENIAC operation he dreamed of an ingenious analog device to implement studies in neutron transport. He and Percy King build the instrument called '**FERMIAC**' to model neutron transport in various types of nuclear systems.

5 Application in Image Processing

Monte Carlo methods are very useful for simulating phenomena with significant uncertainty in inputs and systems with a large number of coupled degrees of freedom. The method is employed to solve a variety of problems in fields such as physical sciences, engineering, computational biology, computer graphics, applied statistics, artificial intelligence for games, design and visuals, search and rescue, finance and business, integration, simulation and optimization, inverse problems and petroleum reservoir management.

The use of Monte Carlo simulations in diagnostic medical imaging research is widespread due to its flexibility and the relative ease with which many different calculations can be performed. These simulations sometimes span large parameter spaces or obtain estimates of quantities that are not simple to measure empirically, e.g., absorbed dose or x-ray scatter.
The ever-continuing reduction in the cost of computing power has also helped increase the use of this research methodology. The simulations need to be validated before their results can be trusted. This involves generating a simulation that replicates an empirical test and then comparing simulation and empirical results which is itself a challenging task. It is not a simple task to learn how to

program and interpret a Monte Carlo simulation, even when using one of the publicly available code packages. The established computer codes/ simulation programs include EGSnrc, BEAMnrc, Geant4, MCNPX, FLUKA, PENELOPE, GATE, SimSET, MABOSE etc.
In medical physics, the method is extensively used in radiotherapy, diagnostic radiology and nuclear medicine.

5.1 Historical Technique

Twenty years after the pioneering work of Hawthorne and Gardner, Vincze et al. and Fernandez et al., Monte Carlo simulation techniques for X-ray spectroscopy and imaging remains a very active field of research. The ultimate goal of the simulations is the prediction of the response of X-ray imaging and spectroscopy experiments and has been applied successfully to the optimization and design of experiments in silico, dose calculation, the estimation of detection limits, quantification and also as a didactic tool for teachers.

In the 1990s, researchers in this field felt that the “brute force” approach used by general purpose Monte Carlo simulation codes was not appropriate for the needs of the applied X-ray physics community. The computation time for realistic simulations was in most cases unacceptable, in particular when considering the limited computational resources available at the time. On the other hand, variance reduction techniques were considered an ‘art’ used by a relatively restricted group of Monte Carlo code developers. In some cases, such techniques were implemented as code sections that were inserted or modified directly in general purpose Monte Carlo packages. In other cases, specialized simulation tools were developed almost from scratch. At the time, simulations were often prepared and run by the same researchers who developed or expanded the Monte Carlo code.

The past twenty years saw a big effort in standardizing variance reduction techniques for X-ray spectroscopy and imaging applications, inserting them as standard tools, with a well-defined set of definable parameters, in more user-oriented Monte Carlo codes.

5.2 ‘PET’ Technique

Positron Emission Tomography (PET) is a nuclear medicine imaging technique. It is a test that uses a special type of camera and a tracer (radioactive chemical) to look at organs in the body. The tracer usually is a special form of a substance (such as glucose) that collects in cells that are using a lot of energy, such as cancer cells. Positron-emitting radio isotopes which combined with metabolic chemical molecules are injected into the human body for clinical examinations and medical research in vivo.

In modern PET-CT scanners, three dimensional imaging is often accomplished with the aid of a CT X-ray scan performed on the patient during the same session, in the same machine. Monte Carlo method is extensively used to simulate images of various body parts such as the brain and helps in treating tumors.

Similarly there are many applications of the Monte Carlo method in modern day scenarios which are being dealt with around the world. There are elaborate algorithms and computer codes which are put to use for extensive research work in various fields.

References

[1] The Beginning of the Monte Carlo Method by N. Metropolis

[2] Stan Ulam, John von Neumann, and the Monte Carlo Method by Roger Eckhardt

[3] Google.com

Further Reading

[1] Monte Carlo Reference Data Sets for Imaging Research: The Report of AAPM Task Group 195

[2] Simulation of PET Brain Images Using Monte Carlo Method by Sicong YU

[3] Digital mammography image simulation using Monte Carlo by Douglas E. Peplowa and Kuruvilla Verghese

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Rigour in Mathematics

This section introduces advance Mathematics to the readers aiming at high standards of proofs. It stimulates interest and lays the foundation for further studies in different branches.

ANALYTIC HIERARCHY PROCESS: AN OVERVIEW

JYOTI DARBARI

Abstract

Multi-Criteria Decision-Making (MCDM) methods have immense applications in making effective decisions in a variety of areas including planning, selecting a best alternative, resource allocation and resolving conflicts. Among the various MCDM techniques developed, usage of Analytic Hierarchy Process (AHP) has increased exponentially. This paper presents an overview of the methodology adopted in AHP for formulating and analysing decisions and briefly discusses the mathematical background of the process.

Keywords: decision making, criteria, AHP

1 Introduction

A fundamental problem of decision theory is to choose the factors which are important for the decision. Generally, real life situations require complex decisions which are usually characterized by a large number of interacting factors. The problem is how to properly assess the importance of these factors in order to make trade-offs among them; how to derive a system of priorities (weights) that can guide us to make good decisions by choosing a best alternative. To answer the problem of decision making in the face of risk, uncertainty, diversity of factors, and varying opinions and judgments, a multi-criteria decision making approach was developed by Thomas L Saaty [1] -the Analytic Hierarchy Process (AHP) . In the Analytic Hierarchy Process we arrange these factors, once selected, in a hierarchic structure descending from an overall goal to criteria, sub-criteria and alternatives in successive levels

AHP is a multiple criteria decision-making tool based on theory of measurement in a hierarchical structure for dealing with tangible and/or intangible criteria. The basic underlying principle of is to give as much importance to experience and knowledge of people as to the data used while making decisions. The decision maker first identifies his or her main purpose in solving a problem what do we need to do? The alternatives are then generated for the problem in mind what are the possible courses of action? Criteria affecting the evaluation of the alternatives are chosen and weighted according to the priority of their importance to the decision maker. The different alternatives are then evaluated in terms of these criteria, and a best one or best mix is chosen. The alternatives are then potential solutions to the problem

The strength of AHP lies in its simplicity, robustness and its ability to be integrated with other mathematical techniques. It has been extensively used in the theme areas of setting priorities; Generating a set of alternatives; Choosing a best policy alternative ; Determining requirements; Allocating resources; Predicting outcomes (time dependence) risk assessment; Measuring performance; Designing a system; Ensuring system stability ; Optimizing; Planning; Conflict resolution [2].

2 Theory of measurement

The axioms of the AHP theory can be stated as follows [3]:

1. **Axiom : (Reciprocal Comparison)** The strength of the preferences of the DM must satisfy the reciprocal condition i.e. If DMs preference ratio for A to B is x:1 then its preference ratio for B to A 'must be $1 : 1/x$.
2. **Axiom : (Homogeneity)** A bounded scale is utilized for quantifying the preferences.
3. **Axiom : (Independence)**Preferences are stated with the assumption that criteria are independent of the alternatives
4. **Axiom : (Expectations)**Completeness of the hierarchical structure is assumed for the purpose of making a decision.

3 Outline of the Methodology

The AHP model developed consists of four phases, which include structuring the problem to build up the hierarchy, collecting data through pair-wise comparison, determining the priorities, and analysis for the solution of the problem.

3.1 Structuring the problem

In general, a hierarchical model of some societal problem might be one that descends from a focus (an overall objective), down to criteria, down further to sub-criteria which are subdivisions of the criteria and finally to the alternatives from which the choice is to be made.
It is a convenient way to decompose a complex problem in search of cause-effect explanations in steps which form a linear chain and it two purposes. It provides an overall view of the complex relationships inherent in the situation; and helps the decision maker assess whether the issues in each level are of the same order of magnitude, so he can compare such homogeneous elements accurately.
A hierarchy does not need to be complete, that is, an element in a given level does not have to function as an attribute (or criterion) for all the elements in the level below. A hierarchy is not the traditional decision tree. Each level may represent a different cut at the problem.

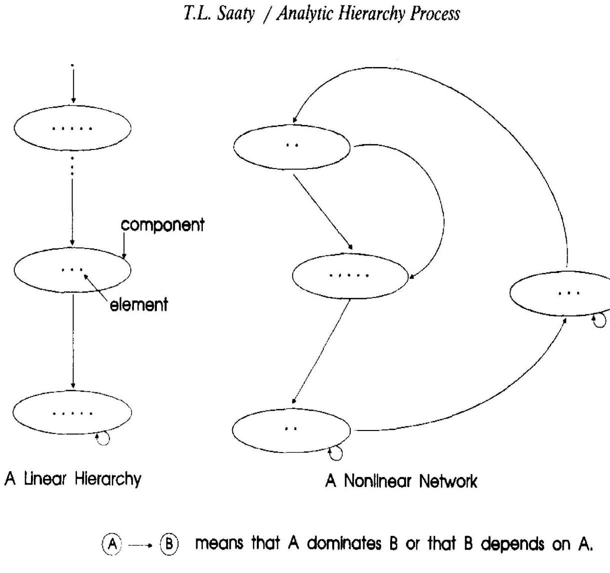

Figure 1: Hierarchy-Linear and non linear [1]

3.2 Generating the Pair-wise Comparison Matrix

AHP helps to incorporate a group consensus. Generally this consists of a questionnaire for comparison of each element (activity) and geometric mean to arrive at a final solution. The judgments are given in the form of paired comparison a_{ij} which indicates the strength with which one i^{th} element dominates j^{th} element as far as the criterion with respect to which they are compared is concerned.
If, for example, the weights are w_i , i = l,..., n, where n is the number of activities, then an entry a_{ij} (is an estimate of w_i/w_j) and the matrix $A = (a_{ij})$, i, j= 1 . . . . . n, has positive entries everywhere and satisfies the reciprocal property $a_{ji} = 1 / a_{ij}$

$$A = \left[a_{ij}\right] = \begin{array}{c} \\ C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & & C_n \end{array} \\ \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \end{array}$$

To measure the strength with respect to a property, AHP uses the method of relative measurements which is useful for properties for which there is no standard scale of measurement (love, political clout). These are known as intangible properties. Since the number of intangible properties is extremely large, devising standard scales for all of them is not feasible and relative scales can be used in such cases. Relative scales are always needed to represent subjective understanding. Measurements in a standard ratio scale can be transformed to measurements in a relative ratio scale by normalizing them.

3.3 Determining the priorities

In the AHP, priorities are synthesized from the second level down by multiplying local priorities by the priority of their corresponding criterion in the level above and adding, for each element in a level according to the criteria it affects. (The second-level elements are multiplied by unity, the weight of the single top-level goal). This gives the composite or global priority of that element, which in turn is used to weight the local priorities of the elements in the level below compared to each other with it as the criterion, and so on to the bottom level. There is an infinite number of ways to derive the vector of priorities from the matrix (a_{ij}). But emphasis on consistency leads to an eigenvalue formulation.

3.4 The measurement of inconsistency

The AHP deals with consistency explicitly because in making paired comparisons, just as in thinking, people do not have the intrinsic logical ability to always be consistent. In a general decision-making environment, we cannot give the precise values of the $\frac{w_i}{w_j}$,but only estimates of them. Thus , if we can identify how serious this inconsistency is and where it can be improved, we can improve the quality of a decision. The eigenvector is associated with the idea of dominance and consistency of judgments. It is the only way to capture inconsistency in the judgments. The degree of inconsistency is measured by the deviation of the principal eigenvalue of the matrix of comparisons from the order of the matrix.

Consistency : If a_{ij} represents the importance of alternative i over alternative j and a_{jk} represents the importance of alternative j over alternative k then a_{ik}, the importance of alternative i over alternative k, must equal $a_{ij}a_{jk}$ for the judgments to be consistent.

Theorem : If A is consistent then principal eigenvalue of A is n.
(For proof refer [4])

It is known from eigenvalue theory, that a small perturbation leads to an eigenvalue problem of the form where is the principal eigenvalue of A where A may no longer be consistent but is still reciprocal. The problem now is: to what extent does w reflect the expert's actual opinion?
For this we need a consistency check [5]. The consistency index (CI), is calculated as - It is the negative average of the other roots of the characteristic polynomial of A. This value is compared with the same index obtained as an average over a large number of reciprocal matrices of the same order whose entries are random. If the ratio (called the consistency ratio CR) of CI to that from random matrices is significantly small (carefully specified to be about 10 or less), we accept the estimate of w. Otherwise, we attempt to improve consistency.

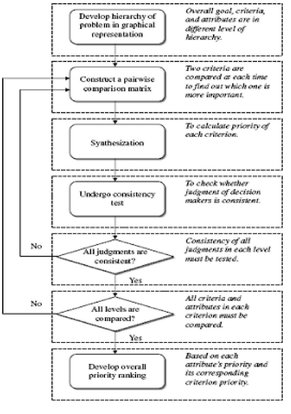

Figure 2: The flow chart of AHP [6]

4 AHP algorithm

In general, the process of judgment in AHP involves making pairwise comparisons between alternatives according to a given criterion. The individual judgments are made by comparing an object, say A, with another, say B, according to the given criterion and these individual judgments are synthesized into a single judgment. The steps involved can be summarized as follows:

1. State the problem.
2. Broaden the objectives of the problem or consider all actors, objectives and its outcome.
3. Identify the criteria that influence the behavior.
4. Structure the problem in a hierarchy of different levels constituting goal, criteria, sub-criteria and alternatives.
5. Compare each element in the corresponding level and calibrate them on the numerical scale. Establish a pairwise comparison decision matrix A whose (i, j)th entry represents the quantified judgment on a pair of n elements. This requires $n(n-1)/2$ comparisons with diagonal elements a_{s1}. A nine-point scale is used for rating where equal, moderate, strong, very strong and extremely important is represented by 1, 3, 5, 7, and 9, respectively. Also, 2, 4, 6 and 8 are used as intermediate values.
6. Normalize the decision matrix and calculate the priorities of the matrix. To do this, divide each element of the matrix by its respective column sum value. The averages of rows of the resultant matrix are the relative priorities.
7. Perform calculations to find the maximum Eigen value, consistency index CI, consistency ratio CR as follows:

$$\mathrm{Aw} = \lambda \mathrm{w}, \mathrm{CI} = \frac{\lambda_{\max} - \mathrm{n}}{\mathrm{n}-1}, \mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} \quad \text{where n is order of A}$$
$$\text{and } \lambda_{\max} = \max\left\{\lambda_{\mathrm{i}}, \mathrm{i} = 1, \ldots.\mathrm{n}\right\}$$

Table 1: **Random index Value (n: size of the matrix)**

	1	2	3	4	5	6	7	8	9
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45

where value of random index (RI) is taken from the following Table1

8. If the maximum Eigen value, CI, and CR(¡0.1) are satisfactory then decision is taken based on the normalized values; else the procedure is repeated till these values lie in a desired range.

References

[1] Saaty, T.L., 1980,*The Analytic Hierarchy Process.*McGraw- Hill, New York.

[2] Vaidya, O.S., Kumar, S., 2006, *Analytic hierarchy process: An overview of applications.* European Journal of Operational Research 169 (1), 129.

[3] Vargas, L.G., *An overview of the Analytic Hierarchy Process and its applications,* European Journal of Operational Research 48/1 (1990) 2-8.

[4] Saaty, T,L.,*How to make a decision:: The Analytic Hierarchy Process,* European Journal of Operational Research 48 (1990) 9-26 9

[5] Saaty, T.L., and Vargas, L.G., *Inconsistency and rank preservation,* Journal of Mathematical Psychology 28/2 (1984).

[6] Ho, W., Dey, P.K., Higson, H.E., 2006. *Multiple criteria decision making techniques in higher education,* International Journal of Educational Management 20 (5), 319337.

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PERTURBED ROBE'S RESTRICTED PROBLEM OF 2+2 BODIES WHEN THE PRIMARIES FORM A ROCHE ELLIPSOID - TRIAXIAL SYSTEM

BHAVNEET KAUR

Abstract

Robe (1977) has investigated a new kind of restricted three-body problem in which one of the primaries of mass m_1 is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1. The smaller primary is a mass point m_2 outside the shell. The third body of mass m_3, supposed moving inside the shell, is a small solid sphere of density ρ_3, with the assumption that the mass and the radius of the third body are infinitesimal. He further assumed that the mass m_2 describes a Keplerian orbit around the mass m_1. He has proved that the centre of the first primary, is the only equilibrium solution for all values of the density parameter K, mass parameter μ, eccentricity parameter e. Further, he has discussed the linear stability of this equilibrium solution. Szebehely (1967) considered the effect of small perturbations in the Coriolis force on the stability of equilibrium points in the classical restricted problem, keeping the centrifugal force constant and proved that the collinear points remain unstable. Shrivastava and Garain (1991) examined the effect of small perturbations in the Coriolis and centrifugal forces on the position of the equilibrium point. Hallan and Rana (2003) studied the effect of perturbations in the Coriolis and centrifugal forces on the location and stability of the equilibrium points in the Robe's circular restricted problem with density parameter having arbitrary value.

Many authors have worked on problem of 2+2 bodies. Bhavneet and Rajiv (2012) extended the Robe's restricted three-body problem to $2+2$ bodies.

1 Statement of the Problem and Equations of Motion

Let the first primary of mass m_1 be a Roche ellipsoid filled with homogeneous incompressible fluid of density ρ_1 and the second primary a triaxial rigid body of mass $m_2 (m_1 > m_2)$ outside the ellipsoid. The two infinitesimal bodies (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4 respectively inside the ellipsoid. Let R denote the distance between the centres of mass of m_1 and m_2 . Let m_2 describe a circular orbit of radius R around m_1 with constant angular velocity $\boldsymbol{\omega}$. The masses m_3 and m_4 mutually attract each other, do not influence the motions of m_1 and m_2 but are influenced by them. Consider a uniformly rotating coordinate system $Ox_1x_2x_3$ with origin of the coordinate system at the centre of the bigger primary ,Ox_1 pointing towards m_2 and Ox_1x_2

being the orbital plane of m_2 around m_1 coinciding with the equatorial plane of m_1. The coordinate system $Ox_1x_2x_3$. Let the synodic system of coordinates initially coincident with the inertial system rotate with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1. Let initially the principal axes of m_2 be parallel to the synodic axes and their axes of symmetry be perpendicular to the plane of motion. Since m_2 is revolving without rotation about m_1 with the same angular velocity as that of the synodic axes, the principal axes of m_2 will remain parallel to them throughout the motion. The perturbations in the Coriolis and centrifugal forces are expressed with the help of parameters α, β, the unperturbed value of each being unity.
We take α and β as

$$\alpha = 1 + \epsilon, |\epsilon| \ll 1,$$
$$\beta = 1 + \epsilon', |\epsilon'| \ll 1.$$

The angular velocity $\boldsymbol{\omega}$ of the triaxial body of mass m_2 is given by

$$\omega^2 = \frac{G\,(m_1 + m_2)}{R^3}\left[1 + \frac{3}{2}\,(2\sigma_1 - \sigma_2)\right]$$

where

$$\sigma_1 = \frac{a^2 - c^2}{5R^2}, \sigma_2 = \frac{b^2 - c^2}{5R^2}, \quad \sigma_i << 1 (i = 1, 2)$$

We fix the units such that $m_1 + m_2 = 1, R = 1$. We choose t in such a way that $G = 1$.
The quantity μ becomes numerically equal to the ratio

$$\frac{m_2}{m_1 + m_2}.$$

In the new units $\omega^2 = 1 + \frac{3}{2}\,(2\sigma_1 - \sigma_2)$.
Hence,the equations of motion of m_3 and m_4 in the Robe's circular restricted problem of $2 + 2$ bodies when the bigger primary is a Roche ellipsoid and the smaller primary a triaxial body in the perturbed system in dimensionless cartesian coordinates are given by

$$\ddot{x}_1^{(i)} - 2\alpha\omega\dot{x}_2^{(i)} = U^{(i)}_{x_1^{(i)}}, \tag{1}$$

$$\ddot{x}_2^{(i)} + 2\alpha\omega\dot{x}_1^{(i)} = U^{(i)}_{x_2^{(i)}}, \tag{2}$$

$$\ddot{x}_3^{(i)} = U^{(i)}_{x_3^{(i)}} \tag{3}$$

where

$$
\begin{aligned}
U^{(i)} =& \frac{\mu_j}{R_{ij}} + D_i\Big[B_i + \mu\left\{\left(x_1^{(i)}\right)^2 - \frac{1}{2}\left(x_2^{(i)}\right)^2 - \frac{1}{2}\left(x_3^{(i)}\right)^2\right\} \\
& + \frac{\beta\omega^2}{2}\left\{\left(x_1^{(i)}\right)^2 + \left(x_2^{(i)}\right)^2\right\} + \frac{\mu}{2}\left(2\sigma_1 - \sigma_2\right)\Big\{6\left(x_1^{(i)}\right)^2 \\
& - \frac{3}{2}\left(x_2^{(i)}\right)^2 - \frac{3}{2}\left(x_3^{(i)}\right)^2\Big\} - \frac{3}{2}\mu\left(\sigma_1 - \sigma_2\right)\left(x_2^{(i)}\right)^2 \\
& - \frac{3}{2}\mu\sigma_1\left(x_3^{(i)}\right)^2 + \frac{\mu}{2}(\mu\omega^2 + (2\sigma_1 - \sigma_2) + 2)\Big],
\end{aligned}
\tag{4}
$$

where

$$
0 < \mu < 1, \sigma_1, \sigma_2 << 1, \mu_j = \frac{m_j}{m_1 + m_2}, D_i = \left(1 - \frac{\rho_1}{\rho_i}\right) \quad i, j = 3, 4; i \neq j. \tag{5}
$$

Here μ, μ_3, μ_4 are mass ratios and $D_i, (i = 3, 4)$ are density parameters of the two infinitesimal masses.
We observe that the triaxiality of the second primary and small perturbations in the Coriolis and centrifugal forces have significant effect on the equations of motion.

2 Equilibrium Solutions

The equilibrium solutions of m_3 and m_4 are given by

$$
U^{(i)}_{x_1^{(i)}} = U^{(i)}_{x_2^{(i)}} = U^{(i)}_{x_3^{(i)}} = 0 \quad (i = 3, 4).
$$

i.e. ,

$$
-\mu_j \frac{\left(x_1^{(i)} - x_1^{(j)}\right)}{R_{ij}^3} + D_i\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)\left(2\sigma_1 - \sigma_2\right) + \epsilon'\right] x_1^{(i)} = 0, \tag{6}
$$

$$
-\mu_j \frac{\left(x_2^{(i)} - x_2^{(j)}\right)}{R_{ij}^3} + D_i\left[(1 - \mu - C_2) + \frac{3}{2}(1 - 2\mu)\left(2\sigma_1 - \sigma_2\right) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right] x_2^{(i)} = 0, \tag{7}
$$

$$
-\mu_j \frac{\left(x_3^{(i)} - x_3^{(j)}\right)}{R_{ij}^3} + D_i\left[-\mu - C_3 - \frac{3}{2}\mu\left(4\sigma_1 - \sigma_2\right)\right] x_3^{(i)} = 0. \tag{8}
$$

with

$$
i, j = 3, 4; i \neq j.
$$

and

$$
\begin{aligned}
C_l &= 2\left(\pi G\rho_1\right) A_l \\
&= 2\left(\frac{\mu}{\mu^*}\right) A_l \left(l = 1, 2, 3\right).
\end{aligned}
$$

and

$$\mu^* = \frac{\mu}{\pi G \rho_1}.$$

2.1 Collinear Equilibrium Solutions

2.1.1 Equilibrium Solutions lying on x_1- axis

The Equations (7),(8) are satisfied with $x_2^{(i)}, x_3^{(i)}; i = 3,4$ equal to zero. Hence we determine $x_1^{(i)}; i = 3,4$ from the Equation (6) by putting $x_2^{(i)}, x_3^{(i)}; i = 3,4$ equal to zero, we get

$$-\mu_4 \frac{\left(x_1^{(3)} - x_1^{(4)}\right)}{|x_1^{(3)} - x_1^{(4)}|^3} + D_i \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right] x_1^{(3)} = 0. \tag{9}$$

$$-\mu_3 \frac{\left(x_1^{(4)} - x_1^{(3)}\right)}{|x_1^{(4)} - x_1^{(3)}|^3} + D_i \left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right] x_1^{(4)} = 0. \tag{10}$$

Multiplying the Equation (9) by μ_3 and (10) by μ_4 and adding, we get

$$x_1^{(4)} = -\lambda x_1^{(3)} \tag{11}$$

where

$$\lambda = \frac{D_3 \mu_3}{D_4 \mu_4} \tag{12}$$

Substituting in the Equation (9) ,we get

$$x_1^{(3)} = \pm \left[\frac{\mu_4}{D_3\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right](1 + \lambda)^2}\right]^{\frac{1}{3}}, \tag{13}$$

and

$$x_1^{(4)} = \mp\lambda \left[\frac{\mu_4}{D_3\left[(1 + 2\mu - C_1) + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2) + \epsilon'\right](1 + \lambda)^2}\right]^{\frac{1}{3}}. \tag{14}$$

Hence $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ are the equilibrium solutions for m_3 and m_4 respectively provided they lie within the Roche's ellipsoid.
Similarly we find the equilibrium solutions lying on x_2- and x_3- axis. There are no non collinear equilibrium solutions of the system.
In all there exist only six equilibrium solutions of the system, provided they lie within Roche ellipsoid.

2.1.2 Stability of Equilibrium Solutions lying on x_1- axis

Let the equilibrium solution $\left(x_1^{(3)},0,0\right)$ and $\left(x_1^{(4)},0,0\right)$ of m_3 and m_4 be displaced to $\left(x_1^{(3)}+\alpha_1^{(3)},\alpha_2^{(3)},\alpha_3^{(3)}\right)$ and $\left(x_1^{(4)}+\alpha_1^{(4)},\alpha_2^{(4)},\alpha_3^{(4)}\right)$.
The variational equations of m_3 and m_4 are

$$\ddot{\alpha_1^{(i)}}-2\alpha\omega\dot{\alpha_2^{(i)}}=\alpha_1^{(i)}p_1^{(i)}\left(\frac{3+\lambda}{1+\lambda}\right) \tag{15}$$

$$\ddot{\alpha_2^{(i)}}+2\alpha\omega\dot{\alpha_1^{(i)}}=\alpha_2^{(i)}\left(\frac{-p_1^{(i)}}{1+\lambda}+p_2^{(i)}\right) \tag{16}$$

$$\ddot{\alpha_3^{(i)}}=\alpha_3^{(i)}\left(\frac{-p_1^{(i)}}{1+\lambda}+p_3^{(i)}\right) \qquad i=3,4 \tag{17}$$

where

$$\begin{aligned}
\lambda &= \frac{D_3\mu_3}{D_4\mu_4},\\
p_1^{(i)} &= D_i\left[(1+2\mu-C_1)+\frac{3}{2}(1+4\mu)\left(2\sigma_1-\sigma_2\right)+\epsilon'\right]\\
p_2^{(i)} &= D_i\left[(1-\mu-C_2)+\frac{3}{2}(1-2\mu)\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\mu\sigma_2+\epsilon'\right]\\
p_3^{(i)} &= D_i\left[\left[-\mu-C_3-\frac{3}{2}\mu\left(4\sigma_1-\sigma_2\right)\right]\right].
\end{aligned}$$

From the Equation (17), we ascertain that the motion of m_3 and m_4 parallel to x_3 axis is stable when

$$\frac{\left[(1+2\mu-C_1)+\frac{3}{2}(1+4\mu)\left(2\sigma_1-\sigma_2\right)+\epsilon'\right]}{\left[-\mu-C_3-\frac{3}{2}\left(4\sigma_1-\sigma_2\right)+\epsilon'\right]}>1+\frac{D_3\mu_3}{D_4\mu_4}$$
$$i=3,4 \text{ in case } \rho_1>\rho_3,\rho_1>\rho_4 \tag{18}$$

and

$$\frac{\left[(1+2\mu-C_1)+\frac{3}{2}(1+4\mu)\left(2\sigma_1-\sigma_2\right)+\epsilon'\right]}{\left[-\mu-C_3-\frac{3}{2}\left(4\sigma_1-\sigma_2\right)+\epsilon'\right]}<1+\frac{D_3\mu_3}{D_4\mu_4}$$
$$i=3,4 \text{ in case } \rho_1<\rho_3,\rho_1<\rho_4 \tag{19}$$

The remaining Equations (15) and (16) admit solutions of the form $\alpha_i^{(j)}=A_i^{(j)}e^{L_jt}, i=1,2,;j=3,4$.
The characteristic equations of m_3 and m_4 are given by

$$\Lambda_i^4+\Lambda_i^2\left(4\omega^2(1+2\epsilon)-\left(Q_1^{(i)}+Q_2^{(i)}\right)\right)+Q_1^{(i)}Q_2^{(i)}=0$$
$$i=3,4 \tag{20}$$

where

$$Q_1^{(i)} = p_1^{(i)} \left(\frac{3+\lambda}{1+\lambda}\right)$$
$$Q_2^{(i)} = p_2^{(i)} - \frac{p_1^{(i)}}{1+\lambda}.$$

Then, $(\Lambda_1^{(i)})^2 + (\Lambda_2^{(i)})^2 = \left(Q_1^{(i)} + Q_2^{(i)}\right) - 4\omega^2(1+2\epsilon)$,
$(\Lambda_1^{(i)})^2(\Lambda_2^{(i)})^2 = Q_1^{(i)}Q_2^{(i)}$.

$$\begin{aligned}(\Lambda_1^{(i)})^2 + (\Lambda_2^{(i)})^2 = {} & D_i\left[(1+2\mu-C_1) + \frac{3}{2}(1+4\mu)\,(2\sigma_1-\sigma_2) + \epsilon'\right]\left(\frac{2+\lambda}{1+\lambda}\right) \\ & + D_i\left[(1-\mu-C_2) + \frac{3}{2}(1-2\mu)\,(2\sigma_1-\sigma_2) + \frac{3}{2}\mu\sigma_2 + \epsilon'\right] \\ & - 4 - 6(2\sigma_1-\sigma_2) - 8\epsilon \end{aligned} \tag{21}$$

$$\begin{aligned}(\Lambda_1^{(i)})^2(\Lambda_2^{(i)})^2 = {} & D_i^2\left(\frac{3+\lambda}{1+\lambda}\right)\Big[(1+2\mu-C_1)\,(1-\mu-C_2) \\ & + \frac{3}{2}(1+4\mu)\,(2\sigma_1-\sigma_2)\,(1-\mu-C_2) \\ & + \left\{\frac{3}{2}(1-2\mu)\,(2\sigma_1-\sigma_2) + \frac{3}{2}\mu\sigma_2\right\}(1+2\mu-C_1) \\ & - \frac{1}{1+\lambda}\{(1+2\mu-C_1)^2 + 3(1+4\mu)\,(2\sigma_1-\sigma_2)\,(1+2\mu-C_1)\} \\ & + \epsilon'\{(2+\mu-C_1-C_2) - \frac{2}{1+\lambda}(1+2\mu-C_1)\}\Big] \end{aligned} \tag{22}$$

The equilibrium solutions $\left(x_1^{(3)}, 0, 0\right)$ and $\left(x_1^{(4)}, 0, 0\right)$ of m_3 and m_4 respectively when the displacement is given in the direction of x_1- axis or x_2- axis are stable if $(\Lambda_1^{(i)})^2$ and $(\Lambda_2^{(i)})^2$ are real and negative or we must have $(\Lambda_1^{(i)})^2 + (\Lambda_2^{(i)})^2 < 0, (\Lambda_1^{(i)})^2(\Lambda_2^{(i)})^2 > 0$. We observe that the condition of stability are influenced by the small perturbations in the Coriolis and centrifugal forces. Similarly we study the stability of equilibrium solutions lying on x_2 and x_3 axis.

3 Conclusion

We observe that the triaxiality of the second primary and small perturbations in the Coriolis and centrifugal forces have significant effect on the equations of motion. When there are small perturbations ϵ and ϵ' in the Coriolis and centrifugal forces respectively, in the Robe's restricted problem of 2+2 bodies, the number of equilibrium solutions are the same as in the problem with no perturbation ([?]), but positions of the equilibrium solutions have changed. The change in the Coriolis force does not affect the positions of the equilibrium solutions. There exist two equilibrium solutions of the system lying each on x_1- axis or x_2- axis or x_3- axis. Hence, there exist six equilibrium solutions of the system, provided they lie within Roche ellipsoid. We observe that there are no other equilibrium solutions except these six. The stability behavior of the equilibrium solution depends on the sign of the small perturbations in the Coriolis and centrifugal forces.

References

[1] [Bhavneet & Rajiv(2013)]BhavneetActa01 Bhavneet Kaur, Rajiv Aggarwal, Robe's restricted problem of 2+2 bodies when the bigger primary is a Roche ellipsoid, Acta Astronautica, Volume 89, August-September 2013, Pages 31-37, ISSN 0094-5765, http://dx.doi.org/10.1016/j.actaastro.2013.03.022.

[2] [Hallan & Rana(2003)]Hallan05 Hallan, P.P. and Rana, N.:2003, 'Effect of Perturbations in the Coriolis and Centrifugal Forces on the Locations and Stability of the Equilibrium Point in the Robe's Circular Problem with Density Parameter Having Arbitrary Value', Indian J. pure appl. Math., 34(7), 1045-1059.

[3] [Bhavneet & Rajiv(2012)]Bhavneet01 Kaur, Bhavneet and Aggarwal, R.:2012 'Robe's Problem: Its Extension to 2+2 Bodies', Astrophysics and Space Science 339, 283-294.

[4] [Robe (1977)]Ref B Robe, H.A.G.:1977, 'A New Kind of Three-Body Problem', Celestial Mechanics 16, 343-351.

[5] [Shrivastava and Garain (1991)]Shri Shrivastava, A. K. and Garain, D.: 1991,'Effect of perturbation on the location of libration point in the Robe restricted problem of three bodies', Celest. Mech. & Dyn. Astr. 51, 67-73.

[6] [Szebehely(1967)]Szeb01 Szebehely V, Peters CF (1967) 'Complete solution of a general problem of three bodies', Astronomical Journal 72:876-883, DOI 10.1086/110355.

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Extension of Course Contents

A great deal of learning happens beyond the formal coursework. This section hence, aims to provide a creative, fertile setting for productive research that goes beyond the confines of classroom, and precincts of syllabi. It strengthens and expands the existing knowledge and adds interests to the course and provides an experience of transformative learning.

APPLICATIONS OF LINEAR ALGEBRA IN ROBOTICS

ESHA SAHA

Abstract

Many industries are now investing in the study and development of robots due to their high levels of efficiency. Robots work faster then humans and maintain their speed for a long period of time, making them extremely popular in many industries like the automobile industry. To make a robot, one requires the knowledge of various disciplines including mathematics. This paper will explore the applications of linear algebra in robotics.
Keywords: Linear Algebra, Robotics, Matrices

1 Introduction

A robot is formally defined by the International Organization for Standardization (ISO) as a re-programmable, multi-functional manipulator designed to move material, parts, tools or specialized devices through variable programmed motions for the performance of variety of tasks. The word "robot" was first used in 1921 in the play *Rossum's Universal Robots* (RUR) written by Czech writer Karel Capek. Originally the word robot can be traced from Czech word '*robota*' meaning 'forced' or 'compulsory labour'.
Before making robots one must keep in mind: a robot must not harm a human being, obey human beings and a robot may take a human's job but may not leave the person jobless. Robots are mainly used in automobile and other manufacturing industries. They are also used in following different ways:



Figure 1 Figure 2 Figure 3 [1]

[1] All figures generated in Word2013 unless mentioned otherwise. Figure 1, Figure 2 and Figure 3 were taken from Google Images.

1. **Mining**: Robots are used to access unworkable mineral seams and reduce human exposure to the harmful environment conditions. This also includes prospecting minerals on the floor of the ocean.

2. **Defence:** The air force, navy and army use mobile firefighters and other devices that can act more swiftly, than people, in an emergency. For example, the *Daksh* robot of DRDO (Defence Research and Development Organisation) in figure 1.

3. **Entertainment:** Robots are widely used in amusement parks, joy rides, etc. Motion platforms are used to seat people wearing 3D glasses and are taken on simulator rides (A robocoaster ride seat in figure 2).

4. **Medical:** With the improvement in technology, medical robots are used in surgeries. For example in laparoscopic surgery as given in figure 3. The goal of the surgical robot is not to replace the surgeon but to assist the surgeon.

There are many more applications of robots. Only few of the common ones were listed above.
A robot generally consists of three subsystems:

1. **A Motion Subsystem:** It is the physical structure of the robot that carries out desired motion (similar to human arms).

2. **A Recognition Subsystem:** It uses various sensors to gather information about the robot itself, any object being acted upon and about the environment.

3. **A Control Subsystem:** It influences the robot's motion to achieve a given task using information provided by the recognition subsystem.

In the upcoming sections, this paper will aim at explaining the different branches of mathematics (other than linear algebra) applied in robotics. After a detailed analysis and study of the applications of linear algebra in robotics, a conclusion will be drawn at the end of the paper.

2 The Mathematics Behind Robotics

Some other disciplines in mathematics that are widely used in robotics are discussed below:

1. **Optimization algorithms:** It is a procedure which is executed iteratively by comparing various solutions till an optimum or satisfactory solution is reached. Many problems in robot task planning and robotic design can be formulated as an optimization problem.

2. **Combinatorics:** Problems, such as motion planning involve analysis of a large, continuous configuration space. It is often approximated by a graph and then discrete techniques are applied. Combinatorial mathematics is used in such cases.

3. **Statistical algorithms:** This approach represents a robot's uncertainty in perception and action selection. As robots are brought amongst people from factories, there is a greater need to cope with uncertainty. In such cases, statistical algorithms have solved problems in the past few years such as problem of mobile robot localization and mapping. Efforts are also there to develop probabilistic robotics.

3 Application of linear algebra

Now we come to the application of linear algebra in a very simple robotic motion in a given plane. We shall discuss two cases:

Case 1: Robot with one link

Consider figure 4. Let there be a motor attached at point O so that the arm OP can move. The coordinates of P will be given by $x = aCos\theta$, $y = aSin\theta$ where $|OP| = a$. It is clear that this arm will not be able to reach any point on the plane other than the ones lying on the boundary of the circle with radius a. This greatly restricts the function of the robotic arm. Hence we go to case 2.

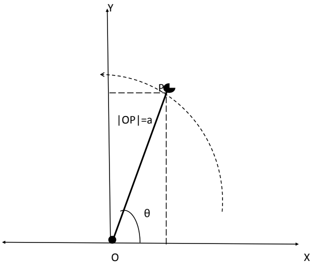

Figure 4: Case 1-One link

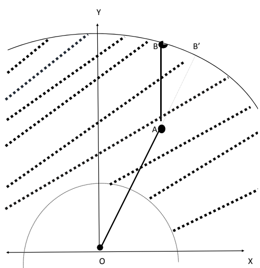

Figure 5: Case 2-two links

Case 2: Robot with two links

Now consider figure 5 where $|OA| > |AB|$. Then point B can reach any point on the plane that is shaded in the figure. To reach any point in this region one needs to apply linear algebra. This process of reaching a point on the plane will be illustrated below.
For this, first we need to determine the coordinates of a point P in XY plane knowing its coordinates in UV plane (which is a rotation of XY plane with angle θ). Consider figure 6.

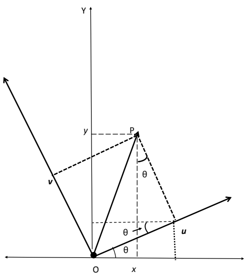

Figure 6: Rotation of angle θ

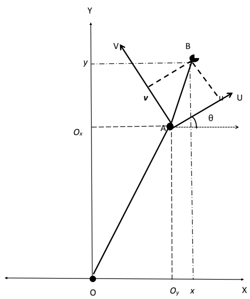

Figure 7: Rotation and transformation

Let the coordinates of P in XY plane be (x, y) and in UV plane be (u, v). If θ is the angle of rotation of XY plane, then

$$x = uCos\theta - vSin\theta \quad (1)$$

$$y = uSin\theta + vCos\theta \quad (2)$$

(1) and (2) can be written as:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \text{ or } [\overrightarrow{p}]_1 = Q[\overrightarrow{p}]_2$$

Here $Q = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix}$ (which is an orthogonal matrix- $Q.Q^T = Q$) is called the rotation matrix, $[\overrightarrow{p'}]_1$ and $[\overrightarrow{p}]_2$ represents coordinates of P in XY plane and UV plane respectively.
Similarly now we determine the coordinates of point B in the XY plane knowing the coordinates in UV plane when both rotation of angle θ and a transformation of $\overrightarrow{O} = O_x\hat{i} + O_y\hat{j}$ is involved.(See figure 7). Then from (1), (2) and figure 7,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Cos\theta & -Sin\theta \\ Sin\theta & Cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} O_x \\ O_y \end{bmatrix}$$

that can also be written as

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} Cos\theta & -Sin\theta & O_x \\ Sin\theta & Cos\theta & O_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

or

$$[\overrightarrow{b}]_1 = T[\overrightarrow{b}]_2 \tag{3}$$

where $[\overrightarrow{b}]_1$ and $[\overrightarrow{b}]_2$ represent the coordinates of point B (in figure 7) in XY plane and UV plane respectively. The matrix T is the homogeneous transformation matrix, where

$$T = \begin{bmatrix} Cos\theta & -Sin\theta & O_x \\ Sin\theta & Cos\theta & O_y \\ 0 & 0 & 1 \end{bmatrix}.$$

Now that we know how to determine the coordinates of a point in XY plane knowing its coordinates in UV plane when both rotation and transformation is involved, we are ready to use the obtained results to move an object from one point to another. For that, consider the given problems-

Problem 1-one link: In figure 8, an object at point A in UV plane has fixed coordinates with $|OP| = a$. Then find the coordinates when the object moves to point B in XY plane if OP moves in angle θ. This can be solved directly by using equation (3). If $[\overrightarrow{a}]_2$ denotes the coordinates of point A in UV plane, then as coordinates of A remain fixed in UV plane, given the angle of rotation θ, we can determine the coordinates of A in XY plane when P moves to B, assume B lies on the boundary of the circle of radius $|OP|$.

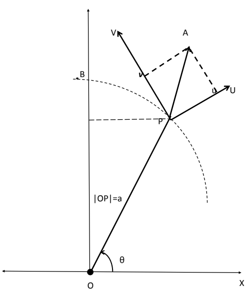

Figure 8: Figure for problem 1

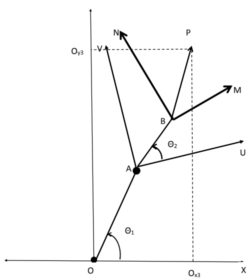

Figure 9: Figure for problem 2

Problem 2-two links: Consider figure with two links. There are three coordinate frames, each with origin at point O (XY plane), A (UV plane) and B (MN plane). Let the point P has fixed coordinates in MN plane denoted by $[p]_3$. From equation (3), $[\overrightarrow{p}]_2 = T_2[\overrightarrow{p}]_3$ and $[\overrightarrow{p}]_1 = T_1[\overrightarrow{p}]_2$,
Then,

$$[\overrightarrow{p}]_1 = T_1T_2[\overrightarrow{p}]_3 = T[\overrightarrow{p}]_3 \tag{4}$$

where $T = T_1T_2$.

To determine T, let $\overrightarrow{OA} = O_{x1}\widehat{i} + O_{y1}\widehat{j}$, $\overrightarrow{AB} = O_{x2}\widehat{i} + O_{y2}\widehat{i}$ and $\overrightarrow{OB} = O_{x3}\widehat{i} + O_{y3}\widehat{j}$

$$\text{Then } T{=}T_1T_2{=}\begin{bmatrix} Cos\theta_1 & -Sin\theta_1 & O_{x1} \\ Sin\theta_1 & Cos\theta_1 & O_{y1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Cos\theta_2 & -Sin\theta_2 & O_{x2} \\ Sin\theta_2 & Cos\theta_2 & O_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} Cos(\theta_1+\theta_2) & -Sin(\theta_1+\theta_2) & O_{x2}Cos\theta_1 - O_{x2}Sin\theta_1 + O_{x1} \\ Sin(\theta_1+\theta_2) & Cos(\theta_1+\theta_2) & O_{x2}Sin\theta_1 + O_{x2}Cos\theta_1 + O_{y1} \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly,

$$O_{x3} = O_{x2}Cos\theta_1 - O_{y2}Sin\theta_1 + O_{x1} \tag{5}$$

and

$$O_{y3} = O_{x2}Sin\theta_1 + O_{y2}Cos\theta_1 + O_{y1} \tag{6}$$

Solving equations (5) and (6) we will get the values of O_{x3} and O_{y3} which will make it possible to find the coordinates of any point given in the MN plane to the XY plane using equation (3).
When three coordinate frames are involved then from equation (4),

$$[\overrightarrow{p}]_1 = T_1T_2[\overrightarrow{p}]_3.$$

If we generalize this result, then for n coordinate frames

$$[\overrightarrow{p}]_1 = T_1T_2.....T_{n-1}[\overrightarrow{p}]_n$$

4 Conclusion

It can be concluded that various areas of mathematics are used in the area of robotics. The study in this paper was of robots with two joints only. However, the robots currently being used in the manufacturing industries have numerous joints as per requirement. Similar calculations can be used to get the results for n number of joints.

References

[1] Saha, S. K., 2008, *Introduction to Robotics*. Tata McGraw-Hill Publishing Company. New Delhi. pp. 1-14, 92-99.

[2] Lay, D.C., 2007, *Linear Algebra and it's Applications*. Pearson Education Asia, Indian Reprint. pp. 246-255.

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AN INTRODUCTION TO QUEUEING THEORY

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Abstract

With the increasing population and ever-increasing demand for goods and services, waiting in line has become a common phenomenon, be it in a restaurant or in a traffic jam. The question that arises is- why is there waiting? This is due to more demand for services than availability. queueing theory attempts to answer this question through mathematical modeling and analysis. The art of applied queueing theory is to construct a model that is simple enough so that it yields to mathematical analysis, yet contains sufficient detail to reflect the behavior of the real system. This paper deals with the queueing theory and its mathematical models.

1 Introduction

queueing theory, also known as theory of overcrowding is the branch of operational research that explores the relationship between demand of the service system and the delays suffered by the users of that system. Generally, queueing theory is concerned with mathematical modeling and analysis of system that provide service to random demands. And more generally, queueing theory is the mathematical theory of queues (waiting lines).

1.1 History

queueing Theory was introduced by Agner Krarup Erlang (1878-1929) in his paper *The Theory of Probabilities and Telephone Conversation* in 1909. His work with the Copenhagen Telephone Company is what prompted his initial foray into the field. He pondered the problem of determining how many telephone circuits were necessary to provide phone service that would prevent customers from waiting too long for an available circuit. In developing a solution to this problem he began to realize that the problem of minimizing waiting time was applicable to many fields and began developing the theory further which laid the path for modern queueing theory.

2 Characteristics of Queueing Process

A queueing system can be described as, customers arriving for service, waiting for service if it is not immediate and if having waited for service then leaving the system after being served. There are six basic characteristics of queueing processes that provide an adequate description of a queueing system. They are given below:

1. **Arrival Pattern of Customers**: In usual queueing situations, the process of arrival is stochastic (stochastic process are systems of events in which the times between events are random variables), and it is thus necessary to know the probability distribution describing the times between successive customer arrivals. It is also necessary to know the reaction of a customer upon entering the system. A customer may decide to wait no matter how long the queue becomes or on the other hand, if the queue is too long, following three situations can occur:

    (a) If a customer decides not to enter the system upon arrival, then the customer is said to have **balked**.

    (b) A customer may enter the queue, but after some time, loses patience and decides to leave. Then the customer is said to have **reneged**.

    (c) A customer may switch from one line to another, and this is called **jockey for position**.

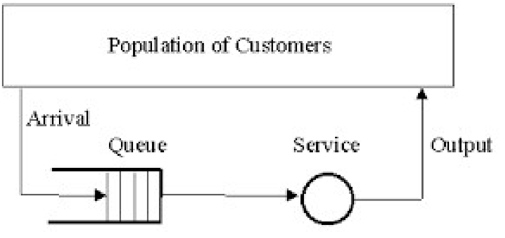
    And an arrival pattern can be of two types- stationary (time-independent) or non-stationary(time-dependent).

2. **Service Patterns**: One generally thinks of one customer being served at a time by a given server, but service may be single or batch. There are many situations where customers may be served simultaneously by the same server, for example, a computer with parallel processing.
The service process may depend on the number of customers waiting for service and this type of situation is referred as **state-dependent service**. Service, like arrivals, can be stationary or non stationary with respect to time. The dependence on time is not to be confused with dependence on state, a queueing system can be both non-stationary or stationary or state dependent.
In general, customers arrive and depart at irregular intervals. Hence, the queue length will assume no definitive pattern unless arrivals and services are deterministic.

3. **Queue Discipline**: It refers to the priority system by which the next customer who is to receive service is selected from a set of waiting customers.
The most common queue discipline that can be observed in everyday life is **First Come, First Served** (FCFS). Some other in common usage are **Last Come, First Served** (LCFS),**Random Selection for Service** (RSS) and a variety of priority schemes, where customers are given priority upon entering the system, the ones with the highest priorities are served first.

4. **System Capacity**: System capacity refers to the overall average number of customers a system can handle in a given space.

5. **Number of Service Channels**: It refers to the number of parallel service stations which can serve customers simultaneously. There are two types of service channel systems:
   (a) Single-Channel system
   (b) Multi-Channel System
6. **Stages of Service**: A queueing system has either a single stage of service or multistage. A physical examination procedure is an example of multistage queueing system and a hair-styling salon is an example of single queueing system.



3 Little's Formulas

Discussion on queueing theory remains incomplete without Little's formula. Little's formulas were developed by John D.C. in the early 1960s and are considered as one of the most powerful relationships in queueing theory. Little related the steady-state mean system sizes to the steady-state average customer waiting times as follows:

Let L be the average number of customers in the queue at any given moment of time (assuming that the steady-state has been reached).
We break L into Lq and Ls, the average number of customers waiting in the queue and the average number of customers in service respectively.
Since customers in the system can only either be in the queue or in service, it goes to show that $L = Lq + Ls$.

Let W be the average time a customer spends in the queueing system. Then Wq and Ws is the average amount of time spent in the queue and the average amount of time spent in service respectively.

As was the similar case before, $W = Wq + Ws$ and all the averages are the steady-state averages.

Let λ be the arrival rate into the system, i.e. the number of customers arriving the system per unit of time, it can be shown that

$$L = \lambda W$$
$$Lq = \lambda Wq$$
$$Ls = \lambda Ws$$

These are known as Little's formulas.

3.1 Kendall's Notation

Kendalls Notation are the notations, widely used to describe elementary queueing systems A/S/m/B/SD where,
A: arrival process
S: service pattern
m: number of servers
B: number of buffers (system capacity)
SD: service discipline (or queue discipline)

For example, the notation M/D/2/100/FCFS indicates a queueing process with exponential inter arrival times, deterministic service pattern, two parallel servers, maximum number of customers allowed is 100,and first- come, first-served is the service discipline.

4 Queueing Mathematical Model

queueing model is used to estimate desired performance measure of the queueing system. It determines the minimum number of servers needed at a service center. It helps in the detection of performance bottleneck or congestion and evaluates alternative system design. There are two types of mathematical models that have been discussed below.

4.1 Single-Channel queueing Model with Poisson Arrivals and Exponential service times (M/M/1):

4.1.1 Assumptions of the Model

The single-channel, single-phase model considered here is one of the most widely used and simplest queueing models. The assumptions required for the modeling is as follows:

1. Arrivals are served on a FCFS basis.
2. Every arrival waits to be served regardless of the length of the line i.e. there is no balking or reneging.
3. Arrivals are independent of preceding arrivals, but the average number of arrivals (the arrival rate) does not change over time.

4. Arrivals are described by a Poisson probability distribution and come from an infinite or very large population.
5. Service time also vary from one passenger to the next and are independent of one another, but their average rate is known.
6. Both the number of items in queue at anytime and the waiting line experienced by a particular item are random variables.
7. Service times occur according to the negative exponential probability distribution.
8. The average service rate is greater than the average arrival rate.
9. The waiting space available for customers in the queue is infinite.

When these seven conditions are met, we can develop a series of equations that define the queue's operative characteristics.

4.1.2 Queueing Equations

Let,
λ = mean number of arrivals per time period (for example, per hour)
μ = mean number of people or items served per time period.

When determining the arrival rate λ and the services rate μ, the same time period must be used. For example, if λ is the average number of arrivals per hour, then μ must indicate the average number that could be served per hour. The queueing equations are as follows:

1. The average number of customers or units in the system Ls i.e. the numbers in the line plus the numbers being served is
$Ls = \frac{\lambda}{(\mu-\lambda)} = \frac{\rho}{(1-\rho)}$
2. The average time a customer spends in the system Ws i.e. the time spent in the line plus the time spent being served is $Ws = \frac{1}{(\mu-\lambda)}$
3. The average number of customers in the queue $Lq = \frac{\lambda^2}{\mu(\mu-\lambda)}$
4. The average time a customers spends waiting in the queue
$Wq = \frac{\lambda}{\mu(\mu-\lambda)}$
5. The utilization factor for the system ρ i.e. the probability that the service facility is being used is $\rho = \frac{\lambda}{\mu}$
6. The present idle time, P_0 i.e. the probability that no one is in the system is $P_0 = 1 - \frac{\lambda}{\mu}$

7. The probability that the number of customers in the system is greater than population size k, $P_n > k$ is $P_n > k = (\frac{\lambda}{\mu})^{k+1})$
8. Length of the non empty queue $Lq' = \frac{\mu}{(\mu-\lambda)}$
9. Probability that waiting time is more than t in the system $= e^{-(\mu-x)}t - (\mu - \lambda)t$
10. Probability that waiting time is more than t in the queue $=e^{-(\mu-\lambda t)} \times \frac{\lambda}{\mu}$

4.2 Multiple-Channel Queueing Model with Poisson Arrivals and Exponential service Times (M/M/S):

The next logical step is to look at a multiple channel queueing system, in which two or more servers or channels are available to handle arriving passengers.

4.2.1 Assumptions of the Model

1. Let us assume that travelers wait for service from one single line and then proceed to the first available server. Each of these channels has an independent and identical exponential service time distribution with mean $\frac{1}{\mu}$. The arrival process is Poisson with rate λ .
2. Arrivals will join a single queue and enter the first available service channel.
3. The multiple-channel system presented here again assumes that arrivals follow a Poisson probability distribution and that service times are distributed exponentially.
4. Service is first come, first served, and all servers are assumed to perform at the same rate.

Other assumptions listed earlier for the single-channel model apply as well.

4.2.2 Queueing Equations

If we let
S = number of channels open,
λ = average arrival rate, and
μ = average service rate at each channel.

The following formulas may be used in the waiting line analysis:

1. Utilization rate is ρ=$\frac{\lambda}{S\mu}$
2. The probability that there are zero customers or units in the system is

$$P_0 = \frac{1}{\sum_{n=0}^{S-1} \frac{(\rho)^n}{n!} + \frac{(\rho)^S}{S!(1-\frac{\rho}{S})}}$$

3. The probability that there are n number of customers in the system

$$P_n = \begin{cases} \frac{(\rho)^n}{n!} f_0 & n < S \\ \frac{\rho^n}{S! S^{(n-S)}} & n \geq S \end{cases}$$

4. Probability that a customer has to wait is $P(n \geq S) = \frac{\mu(\rho)}{(S-1)!(S\mu-\lambda)} \rho_o$

5. The average number of customers or units in the system are

$$Ls = [\frac{(\rho)^S}{\mu S! S(1-\frac{\rho}{S})^2} + \frac{1}{\mu}]\lambda$$

6. The average time a unit spends in the waiting line or being serviced (namely, in the system) is

$$Ws = \frac{(\rho)^S}{\mu S! S(1-\frac{\rho}{S})^2} P_o + \frac{1}{\mu} \ \textbf{\textit{Or}} \ Ws = Ls/\lambda$$

7. The average number of customers or units in line waiting for service

$$Lq = \frac{(\rho)^{S+1}}{S! S(1-\frac{\rho}{S})^2} P_0 \ \textbf{\textit{Or}} \ Lq = Ls - \rho$$

8. The average time a customer or unit spends in the queue waiting for service is

$$Wq == \frac{(\rho)^S}{\mu S! S(1-\frac{\rho}{S})^2} P_0 \ \textbf{\textit{Or}} \ Aq = Ws - \frac{1}{\mu}$$

These equations are obviously more complex than the ones used in the single-channel model, yet they are used in exactly the same fashion and provide the same type of information as the simpler model.

5 Conclusion

The queueing systems are useful throughout society. The capability of these systems can have an important impact on the productivity of the process and the quality of human life. The analysis of queueing models (done in the article) provides fundamental information for successfully designing queueing systems that achieve an appropriate balance between the cost of providing a service and the cost associated with waiting for that service.

References

[1] Donald Gross, Fundamentals of queueing Theory, 3rd edition, John Wiley & Sons (ASIA) Pte Ltd.

[2] https://www.whitman.edu/Documents/Academics/Mathematics/berrryrm.pdf

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Interdisciplinary Aspects of Mathematics

Mathematics is just not a classroom discipline but a tool for organizing and understanding various concepts and applications. This section covers topics that delve into other disciplines, integrating the mode of thinking and knowledge of the respective discipline with Mathematics. The section hence highlights the cosmic scope of Mathematics, leveraging its amalgamation with other disciplines.

ACCESS CONTROL MATRIX

DEEPIKA SAINI and RAJENKI DAS

Abstract

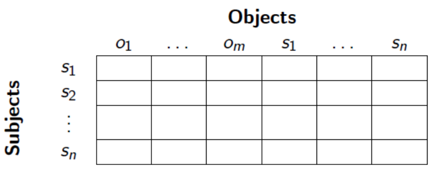
Access Control Matrix is an abstract, formal security model of protection state in computer systems that characterizes the rights of each subject with respect to every object in the system. It was first introduced by Butler W. Lampson in 1971. In today's world, Access Control Matrix has a lot of applications in various fiels like network security and operating systems security. In this paper, we would introduce the topic at a basic level. The paper would throw light on definition, application, implementation, utility. And the paper would end with a final conclusion.

1 Definition

Access Control Matrix is an abstract, formal security model of protection state in computer systems that characterizes the rights of each subject with respect to every object in the system. ACL (Access Control List), with respect to a computer file system, is a list of permissions attached to an object. An ACL specifies which users are granted access to objects, as well as what operations are allowed on given objects.

2 Access Control Model

In access-control model, the entities that can perform actions on the system are called subjects, and the entities representing resources to which access may need to be controlled are called objects.



Example:
The UNIX system defines the rights read, write, and execute. When a process accesses a file, these terms mean what one would expect. When a process accesses a directory, read means to be able to list the contents of the directory;

Objects

Subjects	file 1	file 2	process 1	process 2
process 1	rwo	r	rwxo	w
process 2	a	ro	r	rwxo

$R = \{\text{read, write, execute, own, append}\}$

write means to be able to create, rename, or delete files or subdirectories in that directory and execute means to be able to access files or subdirectories in that directory. When a process accesses another process, read means to be able to receive signals, write means to be able to send signals, and execute means to be able to execute the process as a subprocess.

Access Control models tends to fall into two classes:

- Based on capabilities (The row of access control matrix)
- ACLs i.e. Access Control List (The column of access control matrix)

In an ACL-based model, a subjects access to an object depends on whether its identity appears on a list associated with the object. ACLs have mechanisms to allow access rights to be granted to all the members of a group of subjects. In computer security, general access control covers only access approval, whereby the system makes a decision to grant or reject an access request from an already authenticated subject, based on what the subject is authorized to access. Authentication and access control are often combined, or based on an anonymous access token. Authentication methods and tokens include passwords, biometric scans, physical keys, electronic keys, hidden paths and monitoring by automated systems. Access Control Systems provide the essential services of authorization, identification and authentication, access approval and accountability.

1. authorization specifies what a subject can do.

2. identification and authentication ensure that only legitimate subjects can log on to a system.

3. access approval grants access during operations, by association of users with the resources that they are allowed to access, based on the authorization policy.

3 Implementation

1. **Filesystem:** A filesystem ACL is data structure containing entries that specify individual user or group rights to specific system objects such as programs, processes, or files. These entries are known as access control

entries (ACEs) in the Microsoft Windows NT, OpenVMS and Mac OS X operating systems. In some implementations, an ACE can control whether or not a user, or group of users, may alter the ACL on an object.dern mathematicians.

2. **Networking:** On some type of proprietary computer hardware (in particular routers and switches), an ACL refers to rules that are applied to IP addresses that are available on a host, with a list of network permitted to use the service.

3. **SQL Implementations:** SQL (System Query Language) is a special purpose programming language designed for managing data held in a relational database management system (RDBMS). ACL algorithms have been ported to SQL. Many SQL based systems have used ACL model in their administration modules.

4 Utility

It doesn't define the granularity of protection mechanisms, the ACM can be used as a model of the static access permission in any type of access control system. It doesn't model the rules by which permission can change in any particular system, and therefore only gives an incomplete description of the systems access control security policy. An ACM is an abstract model of permissions at a given point in time.

5 Applications

1. **Social Networks:** In most social networks, such as Facebook and MySpace, some of your personal information can only be accessed by yourself, some can be accessed by your friends, and some can be accessed by everybody. The part of system uses access control.

2. **Web Browsers:** When you browse a web site, and run JavaScript code from that web site, the browser has to control all that JavaScript code can access, and cannot access. For example, a code from one web site cannot access the cookies from another web site, and it cannot modify the contents from another web site either. These controls are conducted by the browsers access control.

3. **Operating Systems:** In an operating system, one user cannot arbitrarily access another users files; a normal user cannot kill another users processes. These are done by operating system access control.

4. **Memory Protection:** In Intel 80x86 architecture, code in one region (for example, in Ring 3), cannot access the data in another more privileged

region (e.g. Ring 0). This is done by the access control implemented in the CPU (e.g. 80386 Protection Mode).

5. **Firewalls:** Firewalls inspect every incoming (sometimes outgoing) packet, if a packet matches with certain conditions, it will be dropped by the firewalls, preventing it from accessing the protected networks. This is also access control.

6 Conclusion

Access Control Matrix plays a major role in safety and security. The access control list organization brings one issue into focus: who can control the modification of access control information. The access control list system was devised to provide more precise control of authority, so some mechanism of exerting that control is needed to provide within the computer an authority structure (self control or hierarchical control).

References

[1] J. H. Saltzer and M. D. Schroeder. The Protection of Information in Computer Systems. In Proceedings of the IEEE, Vol. 63, No. 9. (1975), pp. 1278-1308.

[2] Lampson, Butler W. (1971).protection. Proceedings of the 5 th Princeton Conference of Information Sciences and systems.

[3] CISSP Boot Camp Student Guide, Book 1, Vigilar, Inc.

[4] AccessControlMatrix by Jason Jaskolka

[5] www.wikipedia.com

[6] http://denninginstitute.com/itcore/security/policy.html

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MATHEMATICS AND ARCHITECTURE

PUNEESHA SINGLA

Abstract

Since the inception of the concept of human society, architecture has enjoyed a prominent place in man's life. Over the years, the art of architecture has evolved, and architects and builders have produced an increasing number of buildings with exotic shapes. The design of buildings has always been influenced by mathematical ideas and concepts. Architects have been employing numerous complex mathematical devices like the golden ratio, proportion and symmetry in their works. This research paper will seek to explain this relationship between math and architecture.

1 Introduction

Mathematics and Architecture have been stereotypically defined as two completely diverse studies. But contrary to this, the two have an indispensable relation with each other. Vitruvius, writer of the significant *De Architectura* or *The Ten Books On Architecture*, believed that an architect was a man who had profound knowledge of geometry, in particular. Subsequently, all architects in the Middle Age learnt arithmetic, geometry and aesthetics, as part of their professional courses. Further, the Renaissance Man was primarily expected to study the quadrivium, or the four subjects of arithmetic, geometry, music and astronomy. Another significant reference in this regard is that of Pythagoras. The disciples of the famous Pythagoras were of the opinion that all things are numbers, and this has turned out to be true with respect to architecture too. Therefore, it can be said beyond reasonable doubt that Mathematics and Architecture are so closely woven that they are sometimes indistinguishable.

Why Mathematics in Architecture

It goes without saying that architects use Mathematics for engineering the buildings that they design. From map-making to scale-drawing, the whole process of envisioning and engineering a building, hinges on the focal point of Mathematics.

But mathematical principles are employed to meet four other objectives:

1. **Development Of Harmonious And Beautiful Spacial Forms**: An important objective of making any building remains the portrayal of structures that are in harmony with their surroundings and are pleasing to the human eye. Various mathematical principles are used in order to meet this objective. From the erstwhile Pythagoreans to the modern day Deconstructivism, every movement and belief has been led by the thought of making beautiful buildings whose path is illuminated by Mathematical concepts like symmetry and proportion.

2. **Religious Beliefs**: Religious beliefs too have been known to hold a significant place when it comes to architectural art with buildings being built according to certain sacred beliefs of people of a particular community. The Virupaksha Temple at Hampi, Karnataka, is composed of a cluster of smaller towers about the tallest central tower which represents the holy Mount Kailash, dwelling-place of Lord Shiva, and is believed to depict the endless repetition of universes in Hindu cosmology. Further, the Baptising Font at Pisa, Italy has been built according to the ratio 8:5 with 8 representing the day of resurrection of Jesus Christ and 5 referring to His five wounds. So, all these works show the immense importance attached to a community's religious doctrines in the construction of various structures.

3. **Tessellations**: The concept of drawing tessellations has been comprehensively used since ancient times to create and decorate architectural specimens. This art of repeating geometric patterns creates decorative motifs that give a building a completely distinct look. Such designs are particularly visible in and on Islamic structures. The technique was extensively used in Islamic tiling, minarets and arches. Even today, this art has been used in ensuring that buildings carry a visually impressive look.

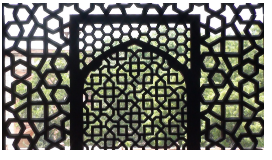

Figure 1: Tessellation in Islamic Architecture

4. **Defence and Environment Goal**: For a building to withstand all sorts of external resistances, it has to be constructed to meet certain defensive and environmental goals. And this is where mathematics plays a crucial role in architecture. Most medieval forts were characterized by star-shaped forts, which ensured that soldiers had better projection points to aim at enemies and that there were fewer dead zones in the fort. The Gherkin in London is the perfect example to show how architects also focus on environmental factors like wind and air currents.

2 Mathematical Principles in Architecture

It is of paramount importance to understand that the association between mathematics and architecture is not limited exclusively to numbers and calculations.

Although numbers do play a crucial role yet they are not the sole point of convergence. On the contrary, it is mathematical concepts, and their theoretical implementation, which forms the backbone of the relationship between the two.

Now, since mathematics forms an inseparable part of architecture, there are a plethora of mathematical principles which are employed by architects. These principles are as follows:

1. **Golden Ratio**: The Golden Ratio has had deep influence on architectural art since time immemorial. It has been one of the most significant principles that not only gives proportion to a building but also makes it aesthetically pleasing. When a line is divided into two parts, golden ratio (denoted by ϕ) is the ratio of the whole length to the longer side such that it is equal to the ratio of the longer side to the smaller side.

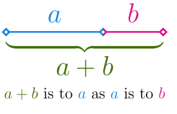

Figure 2

Referring to figure 2,

$$\frac{a+b}{a} = \frac{a}{b} \equiv \phi$$

To find the value of ϕ,

$$\frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\phi}$$

$$\therefore\ \phi = 1 + \frac{1}{\phi}$$

$$\phi^2 - \phi - 1 = 0$$

$$\phi = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$\phi = 1.6180339887\ldots, -0.6180339887\ldots$$

Because ϕ is a ratio between two positive numbers, ϕ is always positive,

$$\phi = 1.6180339887\ldots$$

This ratio has contributed tremendously to the field of architecture. The ratio has been employed in architecture for centuries.

The ratio of the slant height to half the base length of **the Great Pyramids of Egypt** is 1.619, which is almost equal to the value of the golden ratio (as shown in Figure 3). Though the belief that it was purposely built in such manner by yesteryear architects is contentious, it proves that whether it be by accident or on purpose, the golden ratio tends to give the most visually appealing structures.

Although a hugely controversial claim, **The Parthenon in Athens** too, in fact, is said to be completely based on this precious facet of mathematics (as shown in Figure 4). The floor plan of the structure, as well as the placement of columns, is supposed to be designed by using golden rectangles.

The Taj Mahal too displays similar properties in the ratio of the width of its grand central arch to its width, and also in the height of the windows inside the arch to the height of the main section below the domes.

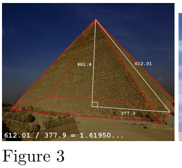

Figure 3

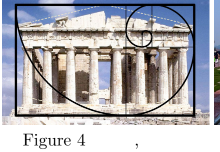
Figure 4 ,

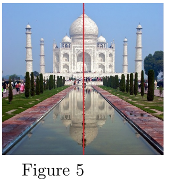
Figure 5

2. **Proportion**

   The concepts of golden ratio and proportion are very closely connected. Proportion is basically defined as the relationship of a part of a structure with another part of it or with the whole of it. But proportion additionally takes into account the relationship between the size of the structure and the size of human body.
   The Parthenon in Athens is an exceptional specimen illustrating proportion. For starters, it has dimensions of 69.5 m, 30.9 m and 13.7 m (length, width and height), which means its length to width ratio is 4:9, exactly the same as height to width. In fact, when the three of them are put together, it gives the ratio 4:6:9 which, as voiced by the Pythagoreans, are based on the harmonies produced by certain musical notes having similar ratios of frequency. Further, the inner area of the building too has 4:9 proportion, and so does the diameter of the outer columns and the distance between their centres.

   This numerical data is the perfect illustration of the immense importance of proportion in yesteryear architecture. It is because of this detailing that the Parthenon is referred to as ‘the most perfect Doric temple’ ever built.

3. **Symmetry**

   Symmetry remains the most fundamental mathematical principle that is being applied in architecture and is spotted in every aspect of architecture, from individual floor plan to the building elements like tiles, doors, windows etc.

   Primarily two types of symmetry are used in the construction of a building.

   Bilateral symmetry is the most common form of symmetry used in architecture in which the two halves of a composition are the mirror images of each other. The Taj Mahal in Agra and the Eiffel Tower in Paris both showcase bilateral symmetry.

   Rotational and Reflectional symmetry is another kind of symmetry in which a part is rotated about a centre point. The Pentagon in Washington DC and the Dome of St. Peters Basilica exhibit rotational symmetry.

Figure 6: The Pentagon exhibits rotational symmetry

   But symmetry as a concept was much more popular in classical architectural compositions. With the advent of the Deconstructivism and its emphasis on non-periodic, distorted and unpredictable structures, symmetry is no longer looked at as a prerequisite for all architectural drawing, as was the case in times gone by.

3 Deconstructivism

Deconstructivism refers to the contemporary form in which the architectural compositions are constructed on the idea of fragmentation and the structures are distorted in a controlled manner to intentionally create discomfort and chaos. A late twentieth century concept, Deconstructivism seeks to disassemble architecture in order to move against the purity and perfection usually associated with it, while also retaining visual appeal. The Imperial War Museum North in Manchester (figure 7) and Walt Disney Concert Hall (figure 8), Los Angeles are famous proponents of Deconstructivism.


Figure 7 Figure 8

Deconstructivism seeks to move away from normal mathematical complexity and pattern-forming, by integrating chaos and dissimilarity in its work. And yet, it is impossible to completely disassociate it from mathematics. The field of study, called Mathematical Chaos, has led to the extensive study of modern day structures which show complete lack of uniformity in shape and design.

4 Other Mathematical References in Architecture

Although it is not possible to refer to all mathematical instances found in architecture, two types of such structures glaringly visible are mentioned below.

1. **Hyperbolic Structures**

   These are architectural works designed by using a hyperboloid in one sheet. They are usually associated with tall towers and buildings. Famous examples include the Shukhov Tower in Polibino and the Vortex Tower in London.

2. **Catenary and Parabolic Structures**

   A catenary refers to a curve that occurs naturally when a chain is allowed to hang under the force of gravity. Catenary is normally associated with parabolic curves, and is found in numerous buildings across the world, including the Kingdom Towers in Riyadh and the Dulles International Airport in Washington D.C.

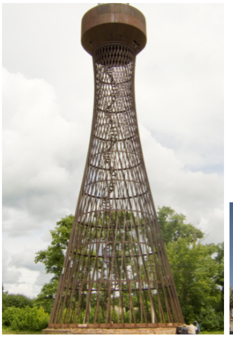

Figure 9 Figure 10

5 Conclusion

There is no doubt as to whether architecture and mathematics bear some sort of relation with each other. Mathematics has had, and continues to have a colossal impact on architectural ideals, and consequently on our environs and surroundings. But the question is, with all the talk of modernisation and the emergence of Deconstructivism, will it continue to be so in times to come? Well, looking at how mathematics has virtually governed how our vicinity looks like for ages, the answer seems to be yes.

References

[1] http://www.emis.de/journals/NNJ/Salingaros.html

[2] www.wikipedia.org

[3] http://www-history.mcs.st-andrews.ac.uk/HistTopics/Architecture.html

[4] http://archive.bridgesmathart.org/2008/bridges2008-47.pdf

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NASH EQUILIBRIUM

BHAVNA PHOGAAT and MANSI YADAV

Abstract

Nash Equilibrium is a concept used in game theory. It involves the study of the optimal outcome of a game wherein no player (in the game) has an incentive to deviate from his/her chosen strategy after considering the opponent's choice. Almost every human interaction- politics, economics, law and religion- can be modeled as a game. Your life is a game if your fate is impacted by the actions of others. In this paper we will highlight the different types of Nash equilibrium while studying various concepts like the battle of the sexes and lobbying.

1 Introduction

Nash Equilibrium is a solution concept of a non-cooperative game involving the interaction of different participants, in which no participant can gain by a unilateral change of strategy if the strategies of the others remain unchanged. Overall, an individual can receive no incremental benefit from changing actions, assuming other players remain constant in their strategies or dormant.

Nash Equilibrium was presented by John Forbes Nash, Jr. who was an American mathematician. John Nash has made fundamental contributions to game theory, differential geometry, and the study of partial differential equations. His theories are used in accounting, economics, computer science, politics, games of skill, evolutionary biology, military theory, artificial intelligence and computing. Nash's theory of non cooperative games was published in 1950 and is known as Nash equilibrium. It provided a conceptually simple but powerful mathematical tool for analyzing a wide range of competitive situations, from corporate rivalries to legislative decision making. In 1994, Nash was awarded the Nobel Prize for his work on game theory.

2 Types of Nash Equilibrium

A game (in strategic or normal form) consists of the following three elements: a set of players, a set of actions (or pure strategies) available to each player and a payoff (or utility) function for each player. The payoff functions represent each player's preferences over action profiles, whereas an action profile is simply a list of actions, one for each player.

A pure-strategy Nash Equilibrium is an action profile with the property that no single player can obtain a higher payoff by deviating unilaterally from this profile.

In some cases, instead of simply choosing an action, players may be able to choose probability distributions over the set of actions available to them. Such randomization over the set of actions are referred to as mixed strategy Nash Equilibrium.

3 Formal Definitions

1. **Nash Eqilibrium**
   Let $B_i(a_{-i}) \in A_i$ be the set of player $i's$ best response actions against $a_{-1} \in A_{-i}$.
   Then,
   $a* = (a*_1, ..., a*_n) \in A$ is a Nash Equilibrium if $a*_i \in B_i(a*_{-i})$ $\forall i \in N$.

2. **Best Response**
   $a_i \in A_i$ is a best response to $a_{-i} \in A_{-i}$ if
   $u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \quad \forall a'_i \in A_i$.

3. **Pure Strategy Nash Equilibrium**
   Given a strategic form game $L = (N, (S_i), (u_i))$, the strategy profiles $s* = (s*_1, s*_2, ..., s*_n)$ is said to be a pure strategy Nash Equilibrium of L if,
   $u_i(s*_i, s*_{-i}) \geq u_i(s_i, s*_{-i}), \quad \forall s_i \in S_i \quad \forall i = 1, 2, ..., n.$

That is, each player's Nash Equilibrium strategy is a best response to the Nash equilibrium strategies of the other players.

4 Pure Strategy Nash Equilibrium

The Battle of Sexes (BOS)

Let a couple agree to meet one evening, but cannot recall if they will be attending the opera or a comedy show. The husband would like to go to the comedy show but the wife would want to go to the opera. Both would prefer to go to the same place rather than different ones. If they do not communicate then their decisions can be represented as the following Payoff matrix.

Let 1 and 2 represent Husband and Wife respectively and let A be the comedy show and B be the Opera.

There are 2 Nash Equilibrium here, namely (A, A) and (B, B). The profile (A, A) is a Nash equilibrium because

$$u_1(M, M) > u_1(F, M) \tag{1}$$

$$u_2(M, M) > u_2(M, F) \tag{2}$$

The profile (F, F) is a Nash Equilibrium because

$$u_1(F, F) > u_1(M, F) \tag{3}$$

$$u_2(F, F) > u_2(F, M) \tag{4}$$

1 \ 2	A	B
A	2,1	0,0
B	0,0	1,2

Table 1: Battle of the sexes

The best response sets are given by:

$$B_1(M) = M; B_1(F) = F; B_2(M); B_2(F) = F \tag{5}$$

Since $M \in B_1(M)$ and $M \in B_2(M), (M, M)$ is a Nash equilibrium. Similarly since $F \in B_1(F)$ and $F \in B_2(F), (F, F)$ is a Nash equilibrium.

5 Mixed Strategy Nash Equilibrium

Consider a (mixed) strategy profile-

$$t = (t_1, t_2, ..., t_n)$$

where t_i is a mixed strategy for player i. The profile σ is a mixed strategy Nash equilibrium if and only if playing t_i is a best response to t_{-i} .That is:

$$u_i(t_i, t_{-i}) \geqslant u_i(s'_i, t_{-i}), \quad \forall s'_i \in S_i \quad \forall i = 1, 2, ..., n. \tag{6}$$

A Lobbying Game

Lobbying is referred to the efforts made to influence decisions made by officials, most often legislators or members of regulatory agencies, in a government.

Suppose two firms simultaneously and independently decide whether to lobby (L) or not lobby (N) the government in hopes of trying to generate favourable legislation. The payoffs of the game is as follows with two pure strategy Nash equilibria:

Does it have a mixed-strategy Nash Equilibrium?

Since this game has only two players and two strategies, this is easy to answer.

Y \ X	L	N
L	-5,-5	25,0
N	0,15	10,10

Table 2: Lobbying Game

Note that both strategies are rational for each player. With only two players and two strategies, a profile of mixed strategies 2 is a Nash equilibrium if and only if:

1. Player 1 is indifferent between L and N when player 2 uses t_2.
2. Player 2 is indifferent between L and N when player 1 uses t_1.

That is, if and only if t_1,t_2 are such that:

$$u_1(L, t_2) = u_1(N, t_2)$$

and

$$u_2(t_1, L) = u_2(t_1, N)$$

Since each player has only two strategies (L and N), any mixed strategy is fully described by

$$t_i = (t_i(L), 1 - t_i(L))$$

Where:

$$t_i(L) = Pr(Player\ i\ chooses\ L)$$

$$1 - t_i(L) = Pr(Player\ i\ chooses\ N)$$

Therefore,

$$u_1(L, t_2) = -5t_2(L) + 25(1 - t_2(L)) = 25 - 30t_2(L)$$

$$u_1(N, t_2) = 0t_2(L) + 10(1 - t_2(L)) = 10 - 10t_2(L)$$

$$u_2(t_1, L) = -5t_1(L) + 15(1 - t_1(L)) = 15 - 20t_1(L)$$

$$u_2(t_1, N) = 0t_1(L) + 10(1 - t_1(L)) = 10 - 10t_1(L)$$

In any mixed-strategy Nash equilibrium, we must have $u_1(L, t_2) = u_1(N, t_2)$. That is:

$$25 - 30t_2(L) = 10 - 10t_2(L)$$

This will be satisfied if:

$$t_2(L) = \frac{3}{4}$$

And we must also have $u_2(t_1, L) = u_2(t_1, N)$.

$$15 - 20t_1(L) = 10 - 10t_1(L)$$

This will be satisfied if:

$$t_1(L) = \frac{1}{2}$$

Therefore, this game has a mixed-strategy equilibrium (t_1, t_2), where:

$$t_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and

$$t_2 = \left(\frac{3}{4}, \frac{1}{4}\right)$$

This example also illustrates that some games may have Nash equilibrium in pure strategies and also in mixed strategies.

6 Strengths and Drawbacks of Nash Equilibrium

6.1 Strengths

- It allows us to make predictions that can be scientifically rejected.
- For the Nash Equilibrium to occur there should be no pre-negotiation prior to the game.
- It can be extended to incorporate repeated games and games with asymmetric information.

6.2 Drawbacks

- It does not always describe the true behavior.
- It implies that individuals have unlimited computing capabilities, but in practice people do not have the ability or resource to compute infinitely and thus make bad decisions.
- Nash Equilibrium only gives the optimum equilibrium value but not the true one.

- Sometimes, there is no Nash Equilibrium at all, making the concept unreliable.
- Many games have many equilibria, and players may not be clear which one to focus on.

7 Conclusion

If a game has a unique Nash Equilibrium and is played among players under certain conditions, only then can the Nash Equilibrium strategy set be adopted. Due to the limited conditions in which Nash Equilibrium can actually be observed, they are rarely treated as a guide to day to day behaviour, or observed in practice in human negotiations.

References

[1] Chapter 2 from An introduction to game theory by Martin J. Osborne, Version:2002/7/23

[2] International Encyclopedia of the Social Sciences, 2nd Edition

[3] www.wikipedia.org

[4] http://lcm.csa.iisc.renet.in/gametheory/In/web-ncp5-purenash

[5] http://oyc.yale.edu/sites/default/files/mixed-strategies-handout

[6] www.siilats.com/docs/economics.htm

[7] www.econ.ucla.edu

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TITANIC SURVIVAL PREDICTION

EISHITA YADAV

Abstract

RMS Titanic was a British passenger ship liner that sank in the Atlantic Ocean in the early morning of 15 April 1912 after colliding with an iceberg during her maiden voyage from Southampton,UK to New York City,US. The twenty lifeboats aboard the ship , were not enough to save a majority of the passengers, which resulted in the loss of 1502 out of 2224 passengers and crew.This paper attempts to describe the analysis,visualizations and the different machine learning techniques used to examine the probability of surviving the Titanic shipwreck using a few dependent variables.

1 Introduction

While the Titanic disaster occurred over 100 years ago,it still attracts researchers looking for understanding as to why some passengers survived and some perished. Using data provided by the Kaggle website, different machine learning techniques will be applied to predict which passengers survived the sinking of the Titanic and the impact of age, sex, passenger class and other attributes on a person's likelihood of surviving the shipwreck. The purpose of this paper is to train a model on the training data set and use this model to predict survivors on the testing dataset. In particular, compare different models like logistic regression, Classification and Regression Trees(CART) and Random Forests.

1.1 Data Set

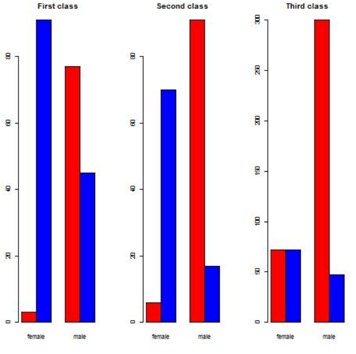
The data used in this paper was taken from the Kaggle website. A training set with 891 passenger samples and a test set with 491 observations was provided. The information provided for each passenger includes their passenger Ids, name, sex, age, passenger class, the number of siblings, spouses aboard, parents-children aboard, ticket details, cabin number, fare of the ticket and port of embarkation. An important thing to note is that only the training set contains the dependent variable 'Survived'. As the variable is missing in the testing set, the true model performance cannot be assessed unless a submission is made on Kaggle.

2 Data Exploration and Preparation

Data exploration is essential to mine data, as it is not possible to get effective results without having enough knowledge about the data. In order to clean the data set, different pre-processing techniques have to used to achieve a clean data set.

Null Value Detection
Null value detection is important for visualizing the structure of the missing data and in selection of an appropriate imputation strategy. Missing values occur when measurements fail, when analysis results get lost or when measurements do not fulfill some prior knowledge of the data. The matrix plot shown below highlights that nearly 60 of the Cabin column and 20 of the Age column has missing values which need a proper imputation strategy.

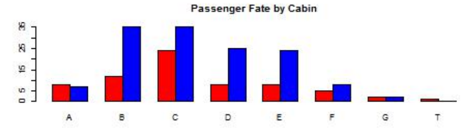
2.1 Exploratory Data Analysis

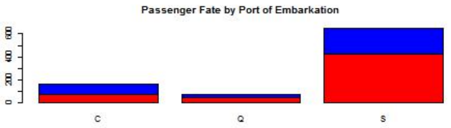


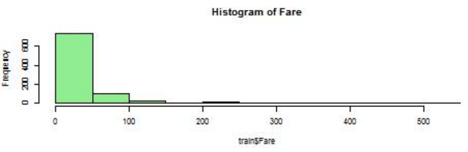
Exploratory data analysis is important as it provides a simple way to obtain a big picture look at the data set and a quick way to check data for mistakes to prevent contamination of subsequent analyses. After building a few plots (given below) we notice a few interesting things.

- In general more women survived than men,this can be related to the "woman and children first policy protocol". Passenger class also played a role in a person's likelihood of survival. First class passengers survived the most and Third Class Passengers the least.
- The fare variable is highly skewed, feature scaling or transformation is required on this variable.

- An additional T cabin value is present which doesn't represent any Cabin.
- The people towards the top more likely survived, because they were closer to the lifeboats.The plot shown below confirms our assumption.People living in B,C,D and E cabins had the highest survival rate.







3 Data Cleaning,Imputation and Feature Creation

For the purpose of Data cleaning , pre-processing and imputation the training set and the testing set were merged . The missing values in the Embarked column were filled in with the most commonly occurring value , Southampton. A new feature 'Title' was created in which we extract the honorific for each observation. Later the title column is cleaned, honorifics like - "Capt", "Don" , "Dona", "Jonkheer", "Rev" were replaced with "unf"(unfortunate) character, because on further inspection people with these honorifics died in the data set.Similarly "Mlle", "Ms", "Mme", "Lady" and "the Countess" was replaced with "Noble" string since people with these titles survived in the data set. Multiple imputation using chained equations is used to fill in the missing observations. The initial letter of the Cabin numbers have been extracted and a blank value has been assigned instead of an NA value. Log Transformation is applied on the fare variable. Finally a binary variable called 'common' was created, which indicates people traveling with the same ticket.

3.1 Splitting into Training, Testing and Validation of Datasets

After cleaning and imputation, we split the merged data set into the train and test sets. The dependent variable Survived is added back to the training set. The training set is further divided into a sub training and a validation set. The model was trained on the sub-training set and later applied on the validation set to gauge its performance. If a reasonable accuracy is achieved then the same model is applied on the test set , the output file is later submitted on Kaggle.

4 Modeling

4.1 Logistic Regression

Logistic regression is a technique that is well suited for examining the relationship between a categorical response variable and one or more categorical or continuous predictor variables. It predicts the probability of the outcome variable being true, in this case it would predict the probability that a person survived. For building the logistic regression model, 10 fold cross validation was used to train the mode and pick the right predictors.

```
ctrl <- trainControl(method = "repeatedcv", number = 10, savePredictions = TRUE)
glm.tune.1 <- train(Survived ~ Pclass+Sex+Age+Embarked+Title+SibSp+Cabin,data=model_train,
                    method="glm",family="binomial",trControl = ctrl)

log_model = glm(Survived~Pclass+Sex+Age+I(Embarked=="S")+Title+SibSp+I(Cabin=="E"),
                data=model_train,family=binomial("logit"))
p = predict(log_model,newdata=validation,type="response")
p<- ifelse(p > 0.5,1,0)
confusionMatrix(p,validation$Survived)
```

Logistic regression gave an accuracy of 0.8531 on the validation set and 0.7655 on the test set.

4.2 CART Model

CART stands for classification and regression trees. It builds a tree by splitting on the values of the independent variables and tries to split the data into subsets of data so that each subset is as pure or homogenous as possible. The standard prediction made by CART is just the majority in each subset. To predict the outcome of an observation or a case you can follow the splits in the tree and in the end you predict the most frequent outcome. Ten fold cross-validation was used to train the CART model, using the varImp() function in R, important predictors were chosen and a cp value of 0.0310219 was used to build the final model.

```
tree2 = rpart(Survived ~ Pclass+Sex+Age+SibSp+Title,data=model_train,method="class",
              cp=.0310219)
predval = predict(tree2,newdata=validation,type="class")
confusionMatrix(predval,validation$Survived)
```

CART gave an accuracy of 0.8362 on the validation set and 0.779 on the test set.

4.3 Random Forests

Similar to the CART model , Random Forests are built to improve the accuracy of CART and it works by building a lot of CART trees. Each tree in the forest is built from a bootstrapped sample of data. To predict the outcome of an observation,each tree in the forest votes for an outcome and we pick the outcome with the majority of votes. The R code used for generation is given

Random Forests gave an accuracy of 0.8305 on the validation set and 0.799 on the test set

5 Conclusion

The table displays the accuracy of different models on the validation and test data set.

	Validation	Test Data
Logistic Regression	0.8531	0.7655
CART model	0.8362	0.779
Random Forests	0.8305	0.799

Random Forests perform the best with an accuracy of 0.799

References

[1] https://www.kaggle.com/c/titanic/forums

[2] http://trevorstephens.com/post/72916401642/titanic-getting-started-with-r

[3] https://www.datacamp.com/courses/kaggle-tutorial-on-machine-learing-the-sinking-of-the-titanic

[4] http://www.r-bloggers.com

[5] http://www.r-bloggers.com

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